10.4 - Iterated Integrals

Let $S \subset \mathbb{R}^2$, $T \subset \mathbb{R}^2$ be rectangles. Then $R = S \times T \subset \mathbb{R}^{4+2}$ is a rectangle also. So $f$ is a bdd function defined on $R$, say $M \leq f(x,y) \leq M$ for all $(x,y) \in R$. Then for all $x \in S$ we have:

$$\int_T f(x,y) \, dy \leq M \int_T f(x,y) \, dy \leq M \int_T f(x,y) \, dy.$$ 

10.4a - Lemma: With notation as above,

$$\int_S \int_T f(x,y) \, dV(x,y) = \int_S \int_T f(x,y) \, dV(x,y) = \int_S \int_T f(x,y) \, dV(x,y).$$

10.4b - Fubini's theorem [10.4.4]:

With notation as above, sps that $f$ is integrable on $R = S \times T$. Then

$$\int_S \int_T f(x,y) \, dV(x,y) = \int_S \int_T f(x,y) \, dV(x,y) = \int_S \int_T f(x,y) \, dV(x,y).$$

Moreover, if $y \mapsto f(x,y)$ is int on $T \forall x \in S$, then $x \mapsto \int_T f(x,y) \, dV(y)$ is int on $S$ and

$$\int_S \int_T f(x,y) \, dV(x,y) = \int_S \int_T f(x,y) \, dV(y) \, dV(x).$$

10.4c - Theorem: With notation as above, sps $f$ int on $S \times T$, $y \mapsto f(x,y)$ int on $T$ $x \in S$ and $x \mapsto f(x,y)$ int on $S$, $y \mapsto \int_T f(x,y) \, dV(y)$ is int on $S$, $y \mapsto \int_T f(x,y) \, dV(x)$ is int on $T$ and

$$\int_S \int_T f(x,y) \, dV(x,y) = \int_S \int_T f(x,y) \, dV(y) \, dV(x) = \int_S \int_T f(x,y) \, dV(x) \, dV(y).$$

10.4d - Fact [10.2.11]: Let $K \subset \mathbb{R}^d$ be compact and let $f : K \to \mathbb{R}$ be cts. Then $G(f) = \{ (x, f(x)) : x \in K \} < \mathbb{R}^{d+1}$ has vol. 0.

10.4e - Fact: Let $B \subset \mathbb{R}^d$ be a compact Jordan set and let $\psi, \phi : B \to \mathbb{R}$ be cts $\forall x \in B$, $\psi(x) \leq \phi(x)$ for all $x \in B$, then $A = \{ (x, t) \in \mathbb{R}^{d+1} : x \in B, \psi(x) \leq t \leq \phi(x) \}$ is a compact Jordan region. If $B$ has vol. 0, then $A$ has vol. 0.

10.4f - Cor: With notation as above, sps $f$ is int on $A$ and that $\forall x \in B$, $t \mapsto f(x,t)$ is int. on $[\psi(x), \phi(x)]$. Then $x \mapsto \int_{[\psi(x), \phi(x)]} f(x,t) \, dt$ is int. on $B$ and

$$\int_B f(x,t) \, dV(x,t) = \int_B \left( \int_{[\psi(x), \phi(x)]} f(x,t) \, dt \right) \, dV(x).$$

10.4g - Cor: With notation as above, if $f$ is cts on $A$, $x \mapsto \int_{[\psi(x), \phi(x)]} f(x,t) \, dt$ is cts on $B$ and above cor. applies.

We construct a compact Jordan region as follows: let $A_1 = [\psi_1, \phi_1]$ where $\psi_1 < \phi_1$. Then $A_1 \subset \mathbb{R}$ is Jordan. Construct compact Jordan regions $A_2, \ldots, A_d$ with $A_i \subset \mathbb{R}$, recur. For $1 \leq i \leq d$, let $\psi_i, \phi_i : A_i \to \mathbb{R}$ be cts $\forall x \in A_i$ and set $A_{1 \ldots d} = \{ (x, t) \in \mathbb{R}^{d+1} : x \in A_i, \psi_i(x) \leq t \leq \phi_i(x) \}$. Then by 10.4e, $A_{1 \ldots d}$ is Jordan.

10.4h - Theorem: Let $A \subset \mathbb{R}^d$ be as above and let $f : A \to \mathbb{R}$ be cts. Then

$$\int_A f(x) \, dV(x) = \int_{A_1} f(x_1) \, dV(x_1) \int_{A_2} f(x_1, x_2) \, dV(x_2) \ldots \int_{A_d} f(x_1, \ldots, x_d) \, dV(x_d).$$

10.4i - Example: Let $E \subset \mathbb{R}^3$ be the compact region held by the coed planes and the plane $x + y + z = 1$. Show that $E$ is Jordan and that

$$\int_E Z \, dV(x,y,z) = \int_0^1 \int_{1-x}^{1-y} Z \, dx \, dy \, dz = \frac{1}{24}.$$