1. Let $U \subset \mathbb{R}^3$ be open, let $a \in U$ and let $F : U \to \mathbb{R}^2$ be a smooth transformation with component functions $f_1, f_2$. Suppose that $\text{rank} [dF(a)] = 2$. Prove that there is a smooth function $f_3 : U \to \mathbb{R}$ such that the transformation $\bar{F} : U \to \mathbb{R}^3$ where $\bar{F} = (f_1, f_2, f_3)$ has a smooth local inverse near $a$. (Hint: Choose a suitable linear function.)

Since $\text{rank} [dF(a)] = 2$, the two vectors $v_1 := \left( \frac{\partial f_1}{\partial y}, \frac{\partial f_1}{\partial z}, \frac{\partial f_1}{\partial x} \right)$ and $v_2 := \left( \frac{\partial f_2}{\partial y}, \frac{\partial f_2}{\partial z}, \frac{\partial f_2}{\partial x} \right)$ are linearly independent. Choose $v_3 := (a, b, c) \in \mathbb{R}^3$ such that $\{v_1, v_2, v_3\}$ is linearly independent (any vector not in span $\{v_1, v_2\}$ will do). Next define $f_3 : U \to \mathbb{R}$ by $f(x, y, z) := ax + by + cz$ for $(x, y, z) \in U$. Since $f_3$ is linear, it is smooth and $v_3 := \left( \frac{\partial f_3}{\partial y}, \frac{\partial f_3}{\partial z}, \frac{\partial f_3}{\partial x} \right)$. Now with $\bar{F} = (f_1, f_2, f_3)$ as above, each of the components of $\bar{F}$ is smooth, and so $\bar{F}$ is itself also smooth. Moreover, $[d\bar{F}(a)]$, its differential matrix at $a$, has rank three because its rows are linearly independent. Hence, $\det [d\bar{F}(a)] \neq 0$ and so by the Inverse Function Theorem $\bar{F}$ has a smooth local inverse near $a$. □

2. Let $R := [0,1] \times [0,1]$; for $n \geq 1$, let $P_n$ be the partition of $R$ with $x_{j,k} = j/n$ where $0 \leq j \leq n$ and $k = 1, 2$ (note that all the subrectangles are squares with side length $1/n$).

a. Find $L(f, P_n)$ and $U(f, P_n)$ where $f : \mathbb{R}^2 \to \mathbb{R}$ is given by

$$f(x, y) := \begin{cases} 1 & \text{if } x = y, \\ 0 & \text{otherwise.} \end{cases}$$

Show that $f$ is Riemann integrable on $R$ and find $\int_R f(x) \, dv(x)$.

The subrectangles determined by $P_n$ are all of the form $R_{i,j} := \{(x, y) : (i-1)/n \leq x \leq i/n, (j-1)/n \leq y \leq j/n\}$ for $1 \leq i, j \leq n$.

For $i, j$ with $1 \leq i, j \leq n$, set $m_{i,j} := \inf\{f(x, y) : (x, y) \in R_{i,j}\}$ and $M_{i,j} := \sup\{f(x, y) : (x, y) \in R_{i,j}\}$.

Then $m_{i,j} = 0$ for all $i, j$ and $M_{i,j} = 1$ if $i = j$ and $M_{i,j} = 0$ otherwise. Hence, we have

$$L(f, P_n) = \sum_{i,j=1}^n m_{i,j}V(R_{i,j}) = 0, \quad U(f, P_n) = \sum_{i,j=1}^n M_{i,j}V(R_{i,j}) = (n+2(n-1)) \frac{1}{n^2} = \frac{3n-2}{n^2}.$$ 

Therefore, $U(f, P_n) - L(f, P_n) = (3n-2)/n^2 \to 0$ and so by Theorem 10.1.8 $f$ is Riemann integrable on $R$; and since $U(f, P_n) \to 0$, we have $\int_R f(x) \, dv(x) = 0$. □

b. Find $L(g, P_n)$ and $U(g, P_n)$ where $g : \mathbb{R}^2 \to \mathbb{R}$ is given by

$$g(x, y) := \begin{cases} 1 & \text{if } y < x, \\ 0 & \text{otherwise.} \end{cases}$$

Show that $g$ is Riemann integrable on $R$ and find $\int_R g(x) \, dv(x)$.

Let $R_{i,j}$, $M_{i,j}$ and $m_{i,j}$ be as in part (a) above. We have $M_{i,j} = 1$ if $j \leq i$ and $M_{i,j} = 0$ otherwise (if $j > i$ and $(x, y) \in R_{i,j}$, then $y \geq x$ and so $g(x, y) = 0$). Hence,

$$U(g, P_n) = \sum_{i,j=1}^n M_{i,j}V(R_{i,j}) = \frac{n(n+1)}{2} \frac{1}{n^2} = \frac{n+1}{2n}.$$ 

We have $m_{i,j} = 1$ if $j < i-1$ and $m_{i,j} = 0$ otherwise (note that if $i = j$, $(j/n, j/n) \in R_{i,j}$ and $g(j/n, j/n) = 0$).

Hence,

$$L(g, P_n) = \sum_{i,j=1}^n m_{i,j}V(R_{i,j}) = \frac{(n-1)(n-2)}{2} \frac{1}{n^2} = \frac{(n-1)(n-2)}{2n^2}.$$ 

Therefore, $U(f, P_n) - L(f, P_n) = (2n-1)/n^2 \to 0$ and so by Theorem 10.1.8 $f$ is Riemann integrable on $R$; and since $U(f, P_n) = (n+1)/2n \to 1/2$, we have $\int_R f(x) \, dv(x) = 1/2$. □

3. Let $R \subset \mathbb{R}^d$ be a rectangle and let $f$ be a function which is integrable on $R$. Prove that $G(f)$, the graph of $f$, has volume 0 (recall that $G(f) := \{(x, f(x)) \in \mathbb{R}^{d+1} : x \in R\}$).

Let $\varepsilon > 0$. Then since $f$ is integrable there is a partition $P$, which subdivide $R$ into the subrectangles $R_1, \ldots, R_n$, such that $U(f, P) - L(f, P) < \varepsilon$. Recall that $U(f, P) = \sum_{i=1}^n M_i V(R_i)$ and $U(f, P) = \sum_{i=1}^n m_i V(R_i)$ where $M_i = \sup\{f(x) : x \in R_i\}$ and $m_i = \inf\{f(x) : x \in R_i\}$ for $i = 1, \ldots, n$. Observe that for $x \in R_i$ we have $f(x) \in [m_i, M_i]$ and hence $(x, f(x)) \in R_i \times [m_i, M_i]$. Moreover $R'_i = R_i \times [m_i, M_i]$ is a rectangle in $\mathbb{R}^{d+1}$ with $V(R'_i) = (M_i - m_i)V(R_i)$. Finally we have $G(f) \subset \bigcup_{i=1}^n R'_i$ and

$$\sum_{i=1}^n V(R'_i) = \sum_{i=1}^n (M_i - m_i)V(R_i) = U(f, P) - L(f, P) < \varepsilon.$$ 

Since $\varepsilon > 0$ was arbitrary, $G(f)$ has volume 0 by Prop. 10.2g. □
4. Let \( f : \mathbb{R}^2 \to \mathbb{R} \) be defined by \( f(x, y) := x^2 y^2 (x^2 + y^2) \) for \((x, y) \in \mathbb{R}^2\) and let
\[
U := \{(x, y) \in \mathbb{R}^2 : 1 < x^2 - y^2 < 4, 1 < xy < 3, x > 0\}.
\]
Show that Theorem 10.5b applies and use it to compute \( \int_U f(x, y) dV(x, y) \).

**Hint:** Observe that \( \phi : U \to \mathbb{R}^2 \) defined by \( \phi(x, y) = (x^2 - y^2, xy) \) is injective and then apply the Change of Variables Formula (Theorem 10.5b).

Note that \( \phi \) is smooth on \( U \) because both of the first partial derivatives of each component are continuous and
\[
|d\phi(x, y)| = \begin{pmatrix} 2x & -2y \\ y & x \end{pmatrix} \quad \text{and} \quad |\det[d\phi](x, y)| = 2(x^2 + y^2).
\]
Note that \( \phi \) is injective on \( U \), \( \phi(U) = (1, 4) \times (1, 3) \) and \( |\det[d\phi]| = 2(x^2 + y^2) \neq 0 \) for all \((x, y) \in U\). Hence the hypotheses of Theorem 10.5b are satisfied and so we have
\[
\int_U x^2 y^2 (x^2 + y^2) dV(x, y) = \int_U \frac{1}{2}(xy)^2 |\det[d\phi](x, y)| dV(x, y) = \int_{\phi(U)} \frac{1}{2} v^2 dV(u, v)
= \int_1^4 \int_1^3 \frac{1}{2} v^2 dv du = 13.
\]
\( \square \)