1. Suppose that the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent. Prove that
\[
\sum_{n=1}^{\infty} |a_n| \leq \sum_{n=1}^{\infty} a_n.
\]
Since the series is absolutely convergent, it converges. Let $s_n = a_1 + \cdots + a_n$ be the $n$th partial sum of the given series and let $s$ denote its sum. Then by the triangle inequality we have for all $n$
\[
|s_n| = |a_1 + \cdots + a_n| \leq |a_1| + \cdots + |a_n| \leq \sum_{n=1}^{\infty} |a_n|.
\]
Then since $s_n \to s$ we have $|s_n| \to |s|$, and therefore $|s| \leq \sum_{n=1}^{\infty} |a_n|$. \(\square\)

2. Determine whether the given series is absolutely convergent, conditionally convergent or divergent. Prove your answer; be sure to cite any tests you use by name.
\[
\sum_{n=2}^{\infty} \frac{(-1)^n}{n \sqrt{\ln n}}
\]
We first apply the Alternating Series Test (Theorem 6.3b or [6.3.2]) with $a_n = \frac{1}{n \sqrt{\ln n}}$ to show that the given series converges. Since both the sequences $\{n\}$ and $\{\sqrt{\ln x}\}$ are positive, nondecreasing and converge to infinity, the product sequence $\{n \sqrt{\ln x}\}$ has the same properties. Hence, the sequence of reciprocals $\{a_n\}$ is positive, nonincreasing and converges to zero. Thus the hypotheses of the Alternating Series Test (6.3b) are satisfied and we conclude that the given series converges.

Next, we use the Integral Test (Theorem 6.2b or [6.2.1]) to show that the series does not converge absolutely and is therefore conditionally convergent. Let $f(x) = \frac{1}{x \sqrt{\ln x}}$ for $x > 1$ and note that $a_n = f(n)$ for $n \geq 2$. One checks as above that $f$ is positive and nonincreasing. Observe that $F(x) := 2\sqrt{\ln x}$ is an antiderivative of $f$ on $(1, \infty)$. Therefore,
\[
\int_{2}^{\infty} f(x) \, dx = \lim_{T \to \infty} F(T) - F(2) = \lim_{T \to \infty} 2(\sqrt{T} - \sqrt{\ln 2}) = \infty.
\]
Since the improper integral diverges, the series $\sum_{n=1}^{\infty} a_n$ diverges (by 6.2b). Therefore, the given series is conditionally convergent. \(\square\)

3. Determine whether the given series is absolutely convergent, conditionally convergent or divergent. Prove your answer; be sure to cite any tests you use by name.
\[
\sum_{n=0}^{\infty} \frac{(-5)^n n^5}{n!}
\]
We apply the Ratio Test (see 6.2h or [6.2.6]) with $a_n = (-5)^n n^5 / n!$. We have
\[
\frac{a_{n+1}}{a_n} = \frac{5^{n+1} (n+1)^5}{(n+1)!} \cdot \frac{n!}{5^n n^5} = \frac{5(n+1)^5}{(n+1)n^5} = \frac{5}{n+1} \left(1 + \frac{1}{n}\right)^5 \to 0.
\]
Since $r = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = 0 < 1$, the series is absolutely convergent. \(\square\)

4. Let $\{f_n\}$ be a sequence of continuous functions defined on $I := [a,b]$. Suppose that $\{f_n\}$ converges to $f$ uniformly on $I$. Prove that $f$ is integrable on $I$ and that
\[
\int_{a}^{b} f(x) \, dx = \lim_{n \to \infty} \int_{a}^{b} f_n(x) \, dx.
\]
Do not use Exercise 5.2.13.

By Theorem 3.4.4 (see 6.4b), $f$ is continuous on $I$ and therefore it is integrable on $I$. Similarly, $f_n$ is integrable on $I$ for each $n$. Let $\varepsilon > 0$, then since $\{f_n\}$ converges to $f$ uniformly on $I$, there is $N > 0$ such that $|f(x) - f_n(x)| < \frac{\varepsilon}{2(b-a)}$ for all $n > N$. Hence for all $n > N$ we have
\[
\left| \int_{a}^{b} f(x) \, dx - \int_{a}^{b} f_n(x) \, dx \right| \leq \int_{a}^{b} |f(x) - f_n(x)| \, dx \leq \frac{\varepsilon}{2(b-a)} (b-a) < \varepsilon
\]
by Theorem 5.2.6 and Corollary 5.2.5. The desired result now follows. \(\square\)
5. Consider the infinite series of functions given below.

\[ \sum_{n=1}^{\infty} \frac{n \sin nx}{2^n} \]

a. Prove that the infinite series converges uniformly on \( \mathbb{R} \) to a continuous function \( f \).

We apply the Weierstrass \( M \)-Test with \( M_n := \frac{n}{2^n} \). Since \( |\sin nx| \leq 1 \) for all \( n \in \mathbb{N} \) and \( x \in \mathbb{R} \) we have

\[ \left| \frac{n \sin nx}{2^n} \right| \leq \frac{n}{2^n} = M_n \quad \text{for all } n \in \mathbb{N}, x \in \mathbb{R}. \]

We show that \( \sum_{n=1}^{\infty} M_n \) converges by the ratio test. We have

\[ \left| \frac{M_{n+1}}{M_n} \right| = \frac{n + 1}{2^{n+1}} \cdot \frac{2^n}{n} = \frac{1}{2} \left( 1 + \frac{1}{n} \right) \to \frac{1}{2} < 1; \]

hence, \( \sum_{n=1}^{\infty} M_n \) converges and therefore the infinite series converges uniformly. Since the function \( \frac{n \sin nx}{2^n} \) is continuous for each \( n \), the limit must be continuous by Theorem 6.4.2. \( \square \)

b. Show that \( f \) is integrable on \([0, \pi]\) and that

\[ \int_{0}^{\pi} f(x) \, dx = \frac{4}{3}. \]

Since \( f \) is continuous on \([0, \pi]\) it is integrable on \([0, \pi]\). By Theorem 6.4.3 we have

\[ \int_{0}^{\pi} f(x) \, dx = \sum_{n=1}^{\infty} \int_{0}^{\pi} \frac{n \sin nx}{2^n} \, dx = \sum_{n=1}^{\infty} \frac{1}{2^n} \cos nx \bigg|_{0}^{\pi} = -\sum_{n=1}^{\infty} \frac{1}{2^n}((-1)^n - 1) = \sum_{n=0}^{\infty} \frac{1}{4^n} = \frac{4}{3}; \]

since only the odd terms of the sum are nonzero, we have reindexed. \( \square \)