1. Let \( \{a_n\}_{n=0}^\infty \) be a nonnegative nonincreasing sequence such that \( a_n \to 0 \). Use the Alternating Series Test (6.3b) to prove that the following power series converges uniformly on \( I = [0,1] \) to a continuous function on \( I \).

\[
\sum_{n=0}^\infty (-1)^n a_n x^n
\]

Let \( \varepsilon > 0 \). Then since \( \{a_n\}_{n=0}^\infty \) is a nonnegative nonincreasing sequence such that \( a_n \to 0 \), there is \( N > 0 \) such that \( 0 \leq a_n < \varepsilon \) for all \( n > N \). Now let \( x \in [0,1] \). Then since \( 0 \leq a_{n+1} x^{n+1} \leq a_n x^n \leq a_n \), the sequence \( \{a_n x^n\}_{n=0}^\infty \) is also nonnegative and nonincreasing; moreover, \( a_n x^n \to 0 \). Hence the Alternating Series Test (6.3b) applies and we conclude that the series \( \sum_{n=0}^\infty (-1)^n a_n x^n \) converges. Denote the sum of the series by \( g(x) \) and the \( n \)-th partial sum by \( g_n(x) \). By the last assertion of the Alternating Series Test (6.3b) we have

\[
|g(x) - g_n(x)| \leq a_{n+1} x^{n+1} \leq a_n < \varepsilon
\]

for all \( n > N \). Hence \( \{g_n\}_n \) converges to \( g \) uniformly on \( I \). Since \( g_n \) is continuous for each \( n \), \( g \) must also be continuous (Theorem 3.4.4).

2. Use induction to prove that for all \( x \in (-1,1) \)

\[
\frac{1}{(1-x)^{k+1}} = \sum_{n=0}^\infty \binom{n+k}{k} x^n \quad \text{for all } x \in (-1,1) \text{ and } k \geq 0.
\]

\((\text{Hint: see Theorem 6.4n.})\)

We use induction on \( k = 0, 1, 2, \ldots \). We first check the case \( k = 0 \). We have

\[
\frac{1}{1-x} = \frac{1}{1-x} = \sum_{n=0}^\infty x^n = \sum_{n=0}^\infty \binom{n+0}{0} x^n \quad \text{for all } x \in (-1,1).
\]

We now suppose that the formula holds for some \( k \geq 0 \). Then differentiating termwise (see Theorem 6.4n or [6.4.12]) we have

\[
\frac{k+1}{(1-x)^{k+2}} = \frac{d}{dx} \frac{1}{(1-x)^{k+1}} = \frac{d}{dx} \sum_{n=0}^\infty \binom{n+k}{k} x^n = \sum_{n=1}^\infty \frac{(n+k)\cdots(n+1)}{k!} n x^{n-1}
\]

\[
= \sum_{n=0}^\infty \frac{(n+k+1)\cdots(n+2)(n+1)}{k!} x^n \quad \text{by reindexing}
\]

\[
= (k+1) \sum_{n=0}^\infty \binom{n+k+1}{k+1} x^n
\]

for all \( x \in (-1,1) \). Dividing through by \( k+1 \) yields the desired result.

3. Find the interval of convergence for the following power series.

\[
\sum_{n=1}^\infty \frac{(-1)^n(x+1)^n}{3^n \sqrt{n}}
\]

We first find the radius of convergence. Setting \( c_n := \frac{(-1)^n}{3^n \sqrt{n}} \) we have

\[
\rho := \limsup_n |c_n|^{1/n} = \limsup_n \frac{1}{3^{n/2} n} = \frac{1}{3}.
\]

Hence, \( R = 1/\rho = 3 \) is the radius of convergence. Thus the series converges for all \( x \) such that \( |x+1| < 3 \). We check the endpoints \( x = -4, 2 \). For \( x = -4 \) we have

\[
\sum_{n=1}^\infty \frac{(-1)^n(x+1)^n}{3^n \sqrt{n}} = \sum_{n=1}^\infty \frac{(-1)^n(-4+1)^n}{3^n \sqrt{n}} = \sum_{n=1}^\infty \frac{1}{\sqrt{n}}.
\]

This series diverges since it is a \( p \)-series with \( p = 1/2 \leq 1 \). For \( x = 2 \) we have

\[
\sum_{n=1}^\infty \frac{(-1)^n(x+1)^n}{3^n \sqrt{n}} = \sum_{n=1}^\infty \frac{(-1)^n(2+1)^n}{3^n \sqrt{n}} = \sum_{n=1}^\infty (-1)^n \frac{1}{\sqrt{n}}.
\]

This series converges by the Alternating Series Test since \( \{1/\sqrt{n}\}_n \) is a nonnegative nonincreasing sequence which converges to 0. Therefore, \( D := (-4,2] \) is the set of points \( x \) at which the series converges.
4. Prove that
\[
\ln x = \ln 2 + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(x-2)^n}{n2^n}
\]
for all \(x \in (0, 4)\).

(Hint: Use Ex. 6.5f.)

By Ex. 6.5f we have for all \(|x-2| < 2\)
\[
\frac{1}{x} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+1}} (x-2)^n.
\]

Then by Cor. 6.4m(iii) (or [6.4.10]) we obtain
\[
\ln x - \ln 2 = \int_2^x \frac{1}{t} \, dt = \sum_{n=0}^{\infty} \frac{(-1)^n (x-1)^{n+1}}{n+1} \cdot \frac{1}{n2^n} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(x-2)^n}{n2^n}
\]
for all \(|x-2| < 2\). The desired result now follows.  

\[\square\]

5. Use Taylor’s Theorem to prove that
\[
|\sin x - x + x^3/6| < 10^{-7}
\]
for all \(x \in [-1, 1]\).

We first show that the fourth Taylor polynomial of \(f(x) = \sin x\) at \(a = 0\) is given by \(T_4(x) = x - x^3/6\). We have
\[
\begin{align*}
f(x) &= \sin x & f(0) &= 0 \\
f'(x) &= \cos x & f'(0) &= 1 \\
f''(x) &= -\sin x & f''(0) &= 0 \\
f'''(x) &= -\cos x & f'''(0) &= -1 \\
f^{(4)}(x) &= \sin x & f^{(4)}(0) &= 0
\end{align*}
\]

Hence we have
\[
T_4(x) = \sum_{k=0}^{4} \frac{f^{(k)}(0)}{k!} = x - \frac{1}{3!} x^3 = x - \frac{x^3}{6}.
\]

Then by Taylor’s Theorem 6.5f (or Taylor’s remainder formula) for all \(x \in R\) there is point \(c\) between \(x\) and 0 such that
\[
|\sin x - x + x^3/6| = |f(x) - T_4(x)| = |R_4(x)| = \left| \frac{f^{(5)}(c)}{5!} x^5 \right| \leq \frac{|x|^5}{120}
\]
since \(|f^{(5)}(c)| = \cos c \leq 1\). Therefore for all \(x \in [-1, 1]\), we have \(|x| \leq 10^{-1}\) and so
\[
|\sin x - x + x^3/6| \leq \frac{|x|^5}{120} < 10^{-7}
\]
as required.  

\[\square\]