1. Prove Cor. 6.5h in the notes: Let $f$ be infinitely differentiable on a nonempty open interval $I = (a - r, a + r)$, where $r > 0$. Suppose there is an $M > 0$ such that $|f^{(n)}(x)| \leq M$ for all $x \in I$ and $n = 0, 1, 2, \ldots$. Then the Taylor series of $f$ converges to $f$ on $I$, i.e.

$$ f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x - a)^n \quad \text{for all } x \in I. $$

2. (Pythagoras) Let $x, y \in \mathbb{R}^d$. Prove that $\|x + y\|^2 = \|x\|^2 + \|y\|^2$ iff $x \cdot y = 0$.

3. Prove that the map $x \in \mathbb{R}^d \mapsto \|x\|_\infty$ given by $\|x\|_\infty = \max\{|x_1|, \ldots, |x_n|\}$ (see Def. 7.1.11) is a norm on $\mathbb{R}^d$.

4. Let $\{x_n\}_n$ and $\{y_n\}_n$ be two sequences of real numbers such that $\sum_{n=1}^{\infty} x_n^2$ and $\sum_{n=1}^{\infty} y_n^2$ both converge. Prove that $\sum_{n=1}^{\infty} x_n y_n$ converges absolutely. (Hint: Use the Cauchy-Schwarz inequality and consider the partial sums.)

5. Let $\{x_n\}_n$ be a sequence in $\mathbb{R}^d$ such that $\sum_{n=1}^{\infty} \|x_n\|$ converges and let $s_n := x_1 + \cdots + x_n$. Prove that $\{s_n\}_n$ is a Cauchy sequence in $\mathbb{R}^d$. Does $\{s_n\}_n$ converge?