6.3 - Absolute and Conditional Convergence

There are many series that converge but are not AC. Such a series is said to converge conditionally (CC).

6.3a Def: A series is said to be alternating if successive terms have opposite signs (pos/neg or neg/pos).

Note: An alternating series has the form \( \sum_{n=1}^{\infty} (-1)^{n+1}a_n \) with \( a_n > 0 \) for all \( n \geq 1 \).

6.3b Theorem (Alternating Series Test) [6.3.2]

Let \( a_n, s_n \) be a seq s.t. i) \( a_n, a_n > 0 \)

\[ ii) a_n, a_n \to \infty \]

\[ iii) a_n \to 0 \]

Then \( \sum_{n=1}^{\infty} (-1)^{n+1}a_n \) converges. Moreover, \( 15 - 5s_n \leq s_{n+1} \) for all \( n \).

Pf: We have \( s_n = a_1 - a_2 + a_3 - a_4 + \ldots + (-1)^{n+1}a_n \) and for all \( k \)

\( s_{2k} \leq s_{2k+2} \leq s_{2k+3} \leq s_{2k+1} \).

For \( k = 0 \), set \( s_0 = 0 \). Since \( a_n \to 0 \) for all \( n \) we have for all \( k \geq 0 \)

\[ s_{2k+2} = s_{2k} + (a_{2k+3} - a_{2k+2}) \geq s_{2k} \] and \( s_{2k+3} = s_{2k+2} + a_{2k+3} \geq s_{2k+2} \) and

\[ s_{2k+4} = s_{2k+2} - a_{2k+2} + a_{2k+3} \geq s_{2k+2} - (a_{2k+2} - a_{2k+3}) = s_{2k+2} \].

So the even terms of \( \{s_n\} \) are nondecr. and the odd terms are nonincr. Moreover, each even term is a lower bd for the odd terms and each odd term is an upper bd for the even terms. So both \( s \) seqs converge;

Set \( s^+ = \lim s_{2k+2} \) and \( s^- = \lim s_{2k+3} \) then \( s^+ - s^- = \lim s_{2k+2} - s_{2k+3} = \lim a_{2k+3} = 0 \). Hence, the two limits agree denote this common value by \( s \). We have \( s_n \to s \) and since \( s \) lies between \( s_n \) and \( s_{n+1} \) we have \( 15 - 5s_n \leq s_{n+1} \) for all \( n \).

Note: If a seq. satisfies conditions i), ii), iii) of 6.3b, write \( a_n \to 0 \).

6.3c Ex: Consider \( \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n^p} \). Note that for all \( p > 0 \), \( \frac{1}{n^p} \to 0 \) and hence the series converges by AST (6.3b). By Cor 6.2c the series is AC for \( p > 1 \) and CC for \( 0 < p \leq 1 \).

6.3d Ex: Show that \( \sum_{n=3}^{\infty} (-1)^n \frac{\ln n}{n^3} \) is CC. We first apply the Integral Test to \( \sum_{n=3}^{\infty} \frac{\ln n}{n^3} \). Let \( f(x) = \frac{\ln x}{x^3} \) for \( x > 3 \), we have \( f(x) > 0 \) for \( x > 3 \) and

\[ \lim_{x \to \infty} \frac{\ln x}{x^3} = 0. \]

Note \( f'(x) = \frac{1 - \ln x}{x^2} < 0 \) for all \( x > e \), so \( f \) is nonincr. on \( [3, \infty) \). Thus the Integral Test (6.2b) applies:

\[ \int_{3}^{\infty} \frac{\ln x}{x^3} \, dx = \lim_{t \to \infty} \int_{3}^{t} \frac{\ln x}{x^3} \, dx = \frac{1}{2} \lim_{t \to \infty} \left( \ln t \right)^2 - \left( \ln 3 \right)^2 = \infty \]

Since the improper integral diverges, the series \( \sum_{n=3}^{\infty} \frac{\ln n}{n^3} \) diverges.

Now we verify that AST (6.3b) applies with \( a_n = \frac{\ln n}{n^3} \). We have \( a_n > 0 \) for all \( n > 3 \); moreover, as shown above, we have \( a_n \to 0 \) and, since \( f \) is nonincr. on \( [3, \infty) \), \( a_n \to 0 \) for \( n > 3 \). By a straightforward application of l'Hôpital's Rule \( \left( \ln x \to 0 \right) \) and so \( a_n \to 0 \). Therefore, we may apply AST to conclude that the series converges. Since it is not AC it is CC.
6.3e Remarks:
Here is a key difference between series that are AC and those that are CC.
6.3.5 (ii) If \(\sum_{n=1}^{\infty} a_n\) is AC then any rearrangement of the terms of the series converges to the same sum. (see 6.3h)
6.3.4 (ii) If \(\sum_{n=1}^{\infty} a_n\) is CC, then for any LE IR, there is a rearrangement of the terms of the series that converges to \(L\) (also \(L=\pm \infty\)).
Recall that for \(x \in \text{IR}\), \(x^+ = \{ x : x > 0 \}\) and \(x^- = \{ x : x < 0 \}\).
Observe that \(x = x^+ - x^-\), \(1x_1 = x^+ + x^-\), \(x^+ x^- > 0\) and \(x^+ x^- = 0\).
6.3f Prop: If \(\sum_{n=1}^{\infty} a_n\) is CC, then both \(\sum_{n=1}^{\infty} a_n^+\) and \(\sum_{n=1}^{\infty} a_n^-\) converge.
PF: Since \(\sum_{n=1}^{\infty} a_n\) converges, then \(\sum_{n=1}^{\infty} a_n^+\) also converges.
Moreover, since \(1a_n = a_n^+ + a_n^-\), \(\sum_{n=1}^{\infty} 1a_n\) also converges. But this contradicts the fact that \(\sum_{n=1}^{\infty} a_n\) is CC and thus not AC. Hence, \(\sum_{n=1}^{\infty} a_n^-\) diverges. A similar argument shows that \(\sum_{n=1}^{\infty} a_n^+\) diverges.
6.3g Def: We say that \(\sum_{n=1}^{\infty} b_n\) is a rearrangement of \(\sum_{n=1}^{\infty} a_n\) if there is a bijection \(f: \text{N} \to \text{N}\) st. \(b_n = a_{f(n)}\) for all \(n \in \text{N}\).
6.3h Theorem [6.3.5] If \(\sum_{n=1}^{\infty} a_n\) is AC and \(\sum_{n=1}^{\infty} b_n\) is a rearrangement of \(\sum_{n=1}^{\infty} a_n\). Then \(\sum_{n=1}^{\infty} b_n\) is also AC and \(\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} a_n\).
PF: Set \(S = \sum_{n=1}^{\infty} a_n\), \(s_n = a_1 + \cdots + a_n\) and \(t_n = b_1 + \cdots + b_n\). It suffices to show that \(t_n \to S\) since applying this argument to \(\sum_{n=1}^{\infty} 1a_n\) proves \(\sum_{n=1}^{\infty} a_n\) converges. Let \(\varepsilon > 0\). Then \(\exists N \in \text{N}\) st. \(\sum_{n=N+1}^{\infty} 1a_n < \frac{\varepsilon}{2}\) and \(1s_n < \frac{\varepsilon}{2}\) for all \(n \geq N\).
Then \(\exists M \in \text{N}\) st. \(a_1, \ldots, a_M < \frac{\varepsilon}{2b_1, \ldots, b_M}\) (can take \(M = \text{max} \{ f(i) : 1 \leq i \leq N \}\)).
Then \(\forall m \geq M, \exists q \in \text{N}\) st. \(2b_1, \ldots, b_M < \frac{\varepsilon}{2a_1, \ldots, a_q}\) and \(1t_m - s_n \leq \sum_{n=1}^{\infty} 1a_n\).
Hence \(t_n \to S\) and \(\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} a_n\).

6.3i Prop. [6.3.6]
Let \(\sum_{n=0}^{\infty} a_n\) and \(\sum_{n=0}^{\infty} b_n\) be AC. Then setting \(c_n = a_0b_n + a_1b_{n+1} + \cdots + a_nb_0\), we have \(\sum_{n=0}^{\infty} c_n\) is AC and \(\sum_{n=0}^{\infty} c_n = \left(\sum_{n=0}^{\infty} a_n\right) \left(\sum_{n=0}^{\infty} b_n\right)\).