7.3 Open and Closed Sets

We will need to discuss the topology of $\mathbb{R}^d$ determined by the norm.

7.3a - Def: For $x_0 \in \mathbb{R}^d$, $r > 0$ set $B_r(x_0) = \{ x \in \mathbb{R}^d : \|x - x_0\| < r \}$ (open ball) and $\overline{B}_r(x_0) = \{ x \in \mathbb{R}^d : \|x - x_0\| \leq r \}$ (closed ball).

Note: Let $x_n$ be a seq in $\mathbb{R}^d$ and let $x_0 \in \mathbb{R}^d$. Then $x_n \to x_0$ iff $\forall \epsilon > 0, \exists N > 0$ s.t. $x_n \in B_\epsilon(x_0)$ for all $n > N$.

7.3b - Prop: Let $x_0, x_1 \in \mathbb{R}^d$, $r > 0$ and s.t. $\|x_1 - x_0\| < r$.

Then setting $r_i = r - \|x_i - x_0\|$, we have $B_{r_i}(x_i) \subset B_r(x_0)$.

Pt: Let $x \in B_{r_i}(x_i)$. Then $\|x - x_i\| < r_i = r - \|x_i - x_0\|$; hence we have

$\|x - x_0\| \leq \|x - x_i\| + \|x_i - x_0\| < r$ and so $x \in B_r(x_0)$.

7.3c - Def: Let $U \subset \mathbb{R}^d$. We say that $U$ is open if $\forall x \in U, \exists r > 0$ s.t. $B_r(x) \subset U$.

A set $F$ is said to be closed if its complement $F^c$ is open. Any open set containing a pt. $x$ is called a neighborhood of $x$.

7.3d - Prop [7.3.2] The sets $\mathbb{R}^d$ and $\emptyset$ are both open and closed. Moreover, every open ball is open and every closed ball is closed.

Pt: $x \in \mathbb{R}^d$, $r > 0$, $B_r(x) \subset \mathbb{R}^d$, so $\mathbb{R}^d$ is open and since $\emptyset$ is also open, both $\mathbb{R}^d$ and $\emptyset$ are both closed as well.

Let $U$ be an open ball then $U = B_r(x_0)$ for some $x_0 \in \mathbb{R}^d$ and $r > 0$. Let $y \in B_r(x_0)$ and set $r_i = r - \|x_i - x_0\|$. Then by 7.3b, $B_{r_i}(x_i) \subset B_r(x_0)$. Hence, $U = B_r(x_0)$ is open. Now let $F$ be a closed ball; then $F = B_r(x_0)$ for some $x_0 \in \mathbb{R}^d$ and $r > 0$. Let $x \in F^c$ and observe $\|x - x_i\| > r$.

So $\|x - x_i\| > r > 0$. Let $y \in B_r(x)$. Then since

$\|x - y\| = \|x - x_i + x_i - y\| = \|x_i - y\| + \|x - x_i\| > r + (r - \|x_i - x_0\|) = r$,

we may conclude that $B_r(x_0) \subset F^c$. Therefore, $F^c$ is open and $F$ is closed.

7.3e - Remark: Let $x_0$ be a pt. in an open set $U \subset \mathbb{R}^d$ and let $x_n$ be a seq in $\mathbb{R}^d$ s.t. $x_n \to x_0$. Then $\exists N > 0$ s.t. $x_n \in U$ for all $n > N$.

7.3f - Ex: Prove that $U = \{ (x, y) \in \mathbb{R}^d : x^2 + y^2 > 1 \}$ is open.

Let $x, y \in U$ and set $r = \min \{ x, y \}$. Show that $B_{r}(x, y) \subset U$.

7.3g Remarks [7.3.3]

i) Given an indexed collection $\{ U_i : i \in I \}$ of open sets, the union $\bigcup_{i \in I} U_i$ is open.

ii) Let $U_1, \ldots, U_n$ be open sets. Then $U_1 \cap \ldots \cap U_n$ is open.

iii) Given an indexed collection $\{ F_i : i \in I \}$ of closed sets, the intersection $\bigcap_{i \in I} F_i$ is closed.

iv) Let $F_1, \ldots, F_n$ be closed sets. Then $F_1 \cup \ldots \cup F_n$ is closed.

v) Given $x \in U_i$. We have $x \in U_i$ for some $i$. Since $U_i$ is open, $\exists r > 0$ s.t. $B_r(x) \subset U_i$.

It follows that $B_r(x) \subset U_i$, and thus $U \cup U_i$ is open.

vi) Let $x \in U_1, \ldots, U_n$. $U_1 \cup \ldots \cup U_n$ is open hence $\exists r > 0$ s.t. $B_r(x) \subset U_1, \ldots, U_n$.

Set $r = \min \{ r_1, \ldots, r_n \}$. Then for each $i$, $B_r(x) \subset B_r(x) \subset U_i$. Thus $B_r(x) \subset U_1, \ldots, U_n$ and so $U$ is open.

vii) follows from (i) and (iv) follows from (ii).

7.3h - Def: Let $E \subset \mathbb{R}^d$ [7.3.6]

i) The interior of $E$, denoted $E^o$, is the largest open set contained in $E$.

ii) The closure of $E$, denoted $\overline{E}$, is the smallest closed set contained in $E$.

iii) The boundary of $E$, denoted $\partial E$, is given by $\partial E = \overline{E} \setminus E^o$. 

7.3 continued

7.3i - Prop. Let $E \subset \mathbb{R}^d$
(i) $x \in E^o \iff \exists r > 0 \text{ s.t. } B_r(x) \subset E$.
(ii) $x \in \overline{E} \iff \forall r > 0, E \cap B_r(x) \neq \emptyset$.
(iii) $x \in \overline{E} \iff \forall r > 0, E \cap B_r(x) \neq \emptyset$ and $E^c \cap B_r(x) \neq \emptyset$.
(iv) $E = E^o \cap \overline{E}$.
(v) $E$ is open iff $E = E^o$.
(vi) $E$ is closed iff $E = \overline{E}$.
(vii) $(E^c)^c = (E^o)^c$ and $(E^o)^c = \overline{E}$

Proof (i) Note that $E^o$ is the largest open subset of $E$ and thus can be expressed as the union of all the open subsets of $E$. So if $B_r(x) \subset E$, we have $B_r(x) \subset E^o$ and so $x \in E^o$. Conversely if $x \in E^o$, $\exists r > 0 \text{ s.t. } B_r(x) \subset E$ since $E^o$ is open.
Thus $B_r(x) \subset E$.
(ii) Sps that $x \notin E$. Then $x \in \overline{E}^c$. Since $E$ is closed, $E^c$ is open and so $\exists r > 0 \text{ s.t. } B_r(x) \subset E^c$. Hence $\exists r > 0 \text{ s.t. } B_r(x) \cap E = \emptyset$.
Conversely, sps $\forall r > 0 \text{ s.t. } B_r(x) \cap E = \emptyset$. Then $F = B_r(x)^c$ is a closed set containing $E$ and so $E \subset F$. Since $x \notin F$, we have $x \notin E$.
The other statements are proved in a similar way. The proofs of (v) and (vi) are especially straightforward.

7.3j - Theorem [7.3.10]
Let $E \subset \mathbb{R}^d$ be nonempty. Then $x_0 \in E \iff \exists a \text{ seq } x_n \in E \text{ s.t. } x_n \to x_0$.
So $E$ is closed iff a convergent seq $x_n\in E$, $\lim x_n \in E$.

Proof: Let $x_0 \in E$. Then for all $n$, $\exists x_n \in E \text{ s.t. } \|x_n - x_0\| < \frac{1}{n}$. Then $x_n \to x_0$.
Sps $\exists \text{ seq } x_n \in E \text{ s.t. } x_n \text{ converges, say } x_n \to x_0$. Let $r > 0$;
then $\exists N \text{ st. } \|x_n - x_0\| < r \text{ for all } n > N$. Hence $B_r(x_0) \cap E \neq \emptyset$ and so $x_0 \in \overline{E}$.
The second assertion follows from the first.