Compact sets play a critical role in Analysis (and other areas of math). The Heine-Borel Theorem asserts that a set \( K \subset \mathbb{R}^d \) is bounded iff \( K \) is closed and bounded. The definition is a bit abstract.

7.4a - Def: Let \( E \subset \mathbb{R}^d \) and let \( \mathcal{U} \) be a collection of open sets in \( \mathbb{R}^d \). We say that \( \mathcal{U} \) is an open cover of \( E \) if \( \forall x \in E \), \( \exists U \in \mathcal{U} \) s.t. \( x \in U \). (i.e. \( E \subset \bigcup_{U \in \mathcal{U}} U \)). A subset of \( \mathcal{U} \) which is also a cover is called a subcover and it is called a finite subcover if it is finite.

7.4b - Ex: Let \( E = [0,1] \subset \mathbb{R} \) and \( \mathcal{U} = \{ B_{1/2}(a) : a \in \mathbb{R} \} \). Then \( \mathcal{U} \) is an open cover of \( E \). \( \mathcal{V} = \{ B_{1/3}(a) : a \in \mathbb{R} \} \) is a subcover. Moreover, \( W_1 = \{ B_{1/3}(0), B_{1/3}(1/2), B_{1/3}(1) \} \) and \( W_2 = \{ B_{1/3}(1/4), B_{1/3}(3/4) \} \) are both finite subcovers.

7.4c - Def: Let \( K \subset \mathbb{R}^d \). Then \( K \) is said to be compact if every open cover has a finite subcover. That is, \( K \) is compact if given an open cover \( \mathcal{U} \), there are \( U_1, \ldots, U_n \in \mathcal{U} \) s.t. \( K \subset U_1 \cup \cdots \cup U_n \).

7.4d - Def: A set \( E \subset \mathbb{R}^d \) is said to be bounded if \( \exists M > 0 \) s.t. \( \forall x \in E \), \( \| x \| < M \).

7.4e - Lemma: Suppose \( K \subset \mathbb{R}^d \) is compact. Then \( K \) is closed and bounded.

pf: Observe that \( U = \bigcup_{m \in N} \bigcup_{n \in N} B_m(0) \) is an open cover of \( K \). Since \( K \) is compact, \( \exists m_1, \ldots, m_N \in N \) s.t. \( K \subset \bigcup_{m \in \{m_1, \ldots, m_N\}} B_m(0) \). Set \( r = \max\{m_1, \ldots, m_N\} \).

Then \( B_r(0) \subset B_m(0) \) for \( m = 1, \ldots, N \). So \( K \subset \bigcup_{m = 1}^{N} B_m(0) \) and so \( \| x \| < r \) for all \( x \in K \). Hence, \( K \) is bounded.

Now suppose \( K \) is not closed. Then \( \exists x \in K \) s.t. \( x \notin K \). For \( n \in N \), set \( U_n = B_n(x) \). Then \( K \subset \bigcup_{n \in N} U_n \) and so \( \mathcal{U} = \{ U_n : n \in N \} \) is an open cover of \( K \). Since \( K \) is compact, \( \exists m_1, \ldots, m_N \in N \) s.t. \( K \subset \bigcup_{n = 1}^{N} U_{m_n} \). We may assume \( m_1 < m_2 < \cdots < m_N \). Since \( U_{m_1} \subset U_{m_2} \subset \cdots \subset U_{m_N} \), we have \( K \subset U_{m_N} \).

Hence \( K \cap B_{m_N}(x) = \emptyset \) and so \( x \notin K \). This contradicts our assumption that \( x \in K \) and thus \( K \) is closed.

7.4f - Lemma [7.4.6]: Let \( \{ E_n \}_{n \in N} \) be a seq of nonempty closed and bounded subsets of \( \mathbb{R}^d \) such that \( E_{n+1} \subset E_n \) for all \( n \). Then \( \bigcap_{n=1}^{\infty} E_n \neq \emptyset \).

pf: For each \( n \), choose \( x_n \in E_n \). Then since \( x_n \in E_n \) for all \( n \), and \( E_1 \) is bounded, the seq \( \{ x_n \}_{n \in N} \) is bounded. Hence by BWT (7.2e or [7.2.14]), \( \{ x_n \}_{n \in N} \) has a convergent subseq \( \{ x_{n_k} \}_{k \in N} \). We claim that \( x_0 = \lim x_{n_k} \in E_n \) for each \( n \). Let \( n \in N \); then \( n_k \geq n \) for \( k \) suff large. Since \( E_n \) is closed, \( x_0 \in E_n \). Therefore, \( \bigcap_{n=1}^{\infty} E_n \neq \emptyset \).

7.4g Heine-Borel Theorem [7.4.7]

Let \( K \subset \mathbb{R}^d \). Then \( K \) is compact iff \( K \) is closed and bounded.

pf: \( \Rightarrow \) This implication follows from 7.4e.

\( \Leftarrow \) Suppose that \( K \) is closed and bounded, but not compact. Then there is an open cover of \( K \) with no finite subcover. Since \( K \) is bounded, there is a closed d-cube \( C_0 = [-L/2, L/2]^d \), s.t. \( K \subset C_0 \). We subdivide \( C_0 \) into \( 2^d \) closed cubes of side length \( L/2 \), \( 2C_0^3 \). Each \( K \cap C_0^3 \) is covered by \( U \) and \( \mathcal{V} \) s.t. \( K \cap C_0^3 \) does not have a finite subcover. Denote one such by \( C_1 \). (ctd.)
7.4 continued

Continue inductively to find a nested sequence of closed subcubes \( \mathcal{C}_n \) s.t. \( \mathcal{C}_{n+1} \subset \mathcal{C}_n \), \( \mathcal{C}_n \) has side length \( 1/2^n \) and \( K \cap \mathcal{C}_n \) does not have a finite subcover. By Lemma 7.4f, \( \cap_{n=1}^{\infty} K \cap \mathcal{C}_n \neq \emptyset \) (note \( K \cap \mathcal{C}_{n+1} \subseteq K \cap \mathcal{C}_n \) and \( K \cap \mathcal{C}_n \) is closed). Let \( x \in \cap_{n=1}^{\infty} K \cap \mathcal{C}_n \). Since \( U \) is an open cover of \( K \), \( x \in U \) for some \( U \in U \). Since \( U \) is open, \( \exists r > 0 \) s.t. \( B_r(x) \subset U \). Choose \( n \in \mathbb{N} \) s.t. \( \frac{L}{2^n} < r \). Since \( x \in \mathcal{C}_n \) and \( \|x - y\| \leq \frac{L}{2^n} \) for any \( y \in \mathcal{C}_n \), \( \mathcal{C}_n \subseteq U \). So \( \cap_{n=1}^{\infty} K \cap \mathcal{C}_n \) is a finite subcover of \( K \cap \mathcal{C}_n \). This results in a contradiction and we conclude that \( K \) must be compact. \( \square \)

7.4h Cor: Let \( K \subset \mathbb{R}^d \) be compact and let \( F \subset K \) be closed. Then \( F \) is also compact.

Pf: This follows immediately from HBT (7.4g) and the fact that any subset of a bdd set must also be bdd. \( \square \)

7.4i Cor: Let \( K_1, \ldots, K_n \subset \mathbb{R}^d \) be compact sets. Then \( K_1 \cup \cdots \cup K_n \) and \( K_1 \cap \cdots \cap K_n \) are both compact.

Pf: By HBT (7.4g), \( K_1, \ldots, K_n \) are closed and bdd. By 7.3g, \( K_1 \cup \cdots \cup K_n \) and \( K_1 \cap \cdots \cap K_n \) are closed. Since \( K_1, \ldots, K_n \) are bdd, \( \exists M_1, \ldots, M_n > 0 \) s.t. \( \forall i, x \in K_i, \|x\| \leq M_i \). So if we set \( M = \max \{M_1, \ldots, M_n\} \), then for any \( x \in K_1 \cup \cdots \cup K_n, \|x\| \leq M \) and similarly \( \forall x \in K_1 \cap \cdots \cap K_n, \|x\| \leq M \). So both \( K_1 \cup \cdots \cup K_n \) and \( K_1 \cap \cdots \cap K_n \) are bdd. Hence by HBT (7.4g) both \( K_1 \cup \cdots \cup K_n \) and \( K_1 \cap \cdots \cap K_n \) are compact.

7.4j - Ex: Let \( x \in \mathbb{R}^d \) and \( r > 0 \), then \( B_r(x) \) is compact since it is closed (by 7.3d') and bdd (\( \forall y \in B_r(x), \|y\| \leq \|y - x\| + \|x\| \leq r + \|x\| \)).

7.4k - Theorem: Suppose \( K \subset \mathbb{R}^d \) with \( K \) compact and \( U \) open. Then \( \exists \) open set \( V \) s.t. \( K \subset V \subset \overline{V} \) and \( \overline{V} \) is compact.

Pf: \( \forall x \in K, \exists r > 0 \) s.t. \( B_{2r}(x) \subset U \). Hence \( B_{r}(x) \subset U \). Then setting \( U_x = B_{r}(x) \), \( U = \bigcup_{x \in K} U_x \) is an open cover of \( K \). Since \( K \) is compact, \( \exists x_1, \ldots, x_n \in K \) s.t. \( K \subset U_{x_1} \cup \cdots \cup U_{x_n} \). Set \( V = U_{x_1} \cup \cdots \cup U_{x_n} \). Then since \( U_x = B_{r}(x) \) is compact for each \( x \), the union \( U_{x_1} \cup \cdots \cup U_{x_n} \) is also compact (see 7.4i). Since \( \overline{V} = U_{x_1} \cup \cdots \cup U_{x_n} \) (by Exercise 7.3.9), and \( \overline{U_{x_i}} \subset C \subset U \) for each \( i \), \( \overline{V} \subset C \). \( \square \)