8.2 Properties of Continuous Functions

8.2 a- Def: Let $D \subseteq \mathbb{R}^p$, then $S \subseteq D$ is said to be relatively open (or open relative to $D$) if there is an open set $U$ s.t. $S = U \cap D$. $S$ is said to be relatively closed if $S = \overline{U \cap D}$ where $E \subseteq \mathbb{R}^p$ is closed.

8.2 b- Notes: Let $E \subseteq \mathbb{R}^p$

(i) $E$ is relatively closed in $D$ iff $D - E$ is relatively open in $D$.
(ii) $E$ is relatively open iff $\forall x \in E, \exists r > 0$ s.t. $B(x) \cap D \subseteq E$.

8.2 c- Lemma: Let $D \subseteq \mathbb{R}^p$ and let $F: D \rightarrow \mathbb{R}^q$ be a function. Then $F$ is cts at $a \in D$ iff $\forall x, 0 < \delta$, $B(x) \subseteq B(y) \cap D \subseteq F^{-1}(B(F(y)))$ (i.e., $F(B(x) \cap D) \subseteq B(F(y))$).

8.2 d- Theorem [8.2.1]: Let $D \subseteq \mathbb{R}^p$ and let $F: D \rightarrow \mathbb{R}^q$ be a function. Then $F$ is cts on $D$ iff $\forall$ open set $U \subseteq \mathbb{R}^q$, $F^{-1}(U)$ is rel. open in $D$.

8.2 e- Cor: With notation as above, $F$ is cts on $D$ iff for every closed set $E \subseteq \mathbb{R}^q$, $F^{-1}(E)$ is relatively closed.

8.2 f- Cor: With notation as above, if $D$ is open, $F$ is cts on $D$ iff $F^{-1}(U)$ is open for every open set $U \subseteq \mathbb{R}^q$.

8.2 g- Theorem: Let $K \subseteq \mathbb{R}^p$ be compact and $\forall x \in \mathbb{R}^q$, $F(x)$ is cts. Then $F(K)$ is compact.

8.2 h- Cor: With $K$ as above, $\exists \alpha \in K$ st. $\sup_{x \in K} \|F(x)\| = \|F(\alpha)\|$. We say that $F: D \rightarrow \mathbb{R}^q$ is bounded if $F(D)$ is bounded.

8.2 i- Cor: With $K, F$ as in 8.2g, $F$ is bounded.
8.2 - Continued

8.2j - Lemma: Let $K \subset \mathbb{R}$ be nonempty and compact. Then \( \exists a, b \in K \) s.t. $K \subset [a, b]$. That is, $\sup K, \inf K \in \mathbb{R}$ (or equiv. max $K$ and min $K$ both exist).

Proof: Since $K$ is bdd $b = \sup K, a = \inf K \in \mathbb{R}$. We show that $a \in K$. For each $n \in \mathbb{N}$, \( \exists x_n \in K \) s.t. $a \leq x_n < a + \frac{1}{n}$. Thus $x_n \rightarrow a$ and since $K$ is closed, $a \in K$. The proof that $b = \sup K \in K$ is similar.

8.2k - Theorem [8.2.5]: Let $K \subset \mathbb{R}^p$ be compact and sps $f: K \rightarrow \mathbb{R}$ is cts. Then $\exists x_0, y_0 \in K$ s.t. $\forall x \in K$,

$$f(x) \leq f(x_0) \leq f(y_0).$$

Proof: By 8.2j, $f(K)$ is compact and so by 8.2j, $\exists a, b \in f(K)$ s.t. $a \leq f(x) \leq b$ for all $x \in K$. Since $a, b \in f(K)$, $\exists x_0, y_0 \in K$ s.t. $x_0 \in f^{-1}(a)$ and $b \in f(y_0)$. Then $\forall x \in K$, $f(x) \leq f(x_0) \leq f(y_0)$.

( Skip the subsection on continuity & connectedness on p199 of the text.)

8.2l - Def: Let $D \subset \mathbb{R}^p$ and let $F: D \rightarrow \mathbb{R}^q$ be a function. Then $F$ is said to be uniformly continuous on $D$ if $\forall \varepsilon > 0, \exists \delta > 0$ s.t. $\forall x, y \in D$, $||x - y|| < \delta \Rightarrow ||F(x) - F(y)|| < \varepsilon$.

8.2m - Theorem [8.2.12]: Let $K \subset \mathbb{R}^p$ be compact and sps $F: K \rightarrow \mathbb{R}^q$ is cts. Then $F$ is unif. cts.

Proof: Let $\varepsilon > 0$ and let $x \in K$. Then since $F$ is cts at $x$, $\exists \delta_x > 0$ s.t. $\forall y \in K$, $||x - y|| < \frac{\delta_x}{2} \Rightarrow ||F(x) - F(y)|| < \frac{\varepsilon}{2}$. Set $U_x = B_{\delta_x}(x)$ and observe that $U = \bigcup U_x : x \in K$ is an open cover of $K$. Since $K$ is compact, $\exists x_1, \ldots, x_n \in K$ s.t. $K \subset U_{x_1} \cup \cdots \cup U_{x_n}$. Set $\delta = \min \{\delta_{x_1}, \ldots, \delta_{x_n}\}$.

Now let $x, y \in K$ s.t. $||x - y|| < \delta$. Then $x \in U_{x_i}$ for some $i$ and so $||x - x_i|| < \delta_{x_i}$ and since $||x - y|| < \delta \leq \delta_{x_i}$, $||y - x_i|| \leq ||y - x|| + ||x - x_i|| < 2\delta_{x_i}$. So $||F(x) - F(y)|| \leq ||F(x) - F(x_i)|| + ||F(y) - F(x_i)|| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$.

8.2n - Prop. [8.2.13]: Let $D \subset \mathbb{R}^p$ be bdd and let $F: D \rightarrow \mathbb{R}^q$ be a function. Then $F$ is unif. cts. on $D$ iff $F$ extends to a cts function on $\overline{D}$. 
