9.2 - The Differential

Recall: f: I \rightarrow \mathbb{R} is a function where I \subseteq \mathbb{R} is an open interval, then f is differentiable at a \in I if \exists f'(a) in \mathbb{R} s.t. 
\lim_{h \to 0} \frac{f(a+h) - f(a) - f'(a)h}{h} = 0. 
If we define 
\varepsilon(h) = \lim_{h \to 0} \frac{f(a+h) - f(a) - f'(a)h}{h} \rightarrow \text{then } f'(a) = f(a) + f'(a)\Delta x + \varepsilon(h) \Delta x \text{ where } \lim_{h \to 0} \varepsilon(h) = 0.

So the error in approx is \varepsilon(h) \Delta x.

9.2a - Note: If f: I \subseteq \mathbb{R} st. \lim_{h \to 0} \frac{f(a+h) - f(a) - c \cdot h}{h} = 0, then f is differentiable at a & f'(a) = c.

9.2b - Def: A function F: \mathbb{R}^p \rightarrow \mathbb{R}^q is said to be affine if there is a linear function L: \mathbb{R}^p \rightarrow \mathbb{R}^q \text{ and a vector } v \in \mathbb{R}^q s.t. F(x) = L(x) + v.

So we can say that f: I \rightarrow \mathbb{R} is differentiable at a if f is approx affine near a.

9.2c - Def: Let D \subset \mathbb{R}^p, let F: D \rightarrow \mathbb{R}^q be a function and let a \in D^0. Say F is differentiable at a, if \exists a linear map A (or a linear transform L: \mathbb{R}^p \rightarrow \mathbb{R}^q) st.

\[ \lim_{h \to 0} \frac{1}{\|h\|} (F(a+h) - F(a) - Ah) = 0 \]

9.2d - Note: F is differentiable at a iff h \mapsto F(a+h) is approx affine in the sense that \exists a \in \mathbb{R}^p and \exists \Delta x \in \mathbb{R}^q s.t. F(a+h) = F(a) + \Delta x \cdot h for h \in U st. \lim_{h \to 0} \frac{1}{\|h\|} (F(a+h) - F(a) - \Delta x \cdot h) = 0. (Hence the affine transform is F(a) + \Delta x \cdot h.)

9.2e - Prop: With notation as in 9.2c, if F is differentiable at a, then F is continuous at a.
Proof: Suppose that F is differentiable at a, then as in 9.2d, \|F(a+h) - F(a) - Ah\| = \|A\| \cdot \|h\| \forall x \in D^0. Then \lim_{h \to 0} F(a+h) - F(a) = \lim_{h \to 0} A(x-a) + \lim_{h \to 0} \varepsilon(x-a) \cdot h = 0.

9.2f - Def: Suppose that F is differentiable at a with A as above, then the linear map associated to A, L: \mathbb{R}^p \rightarrow \mathbb{R}^q is called the differential of F at a and is denoted dF(x).

So dF(x): \mathbb{R}^p \rightarrow \mathbb{R}^q \text{ is given by } dF(x)(h) = Ah \text{ for } h \in \mathbb{R}^p.

9.2g - Note: We have \lim_{h \to 0} \frac{1}{\|h\|} (F(a+h) - F(a) - dF(x)(h)) = 0. We will see below that dF(x) is the unique linear map with this property.

9.2h - Prop: Let D \subset \mathbb{R}^p, F: D \rightarrow \mathbb{R}^q, a \in D^0 be as above and let f_1, ..., f_q be the component functions of F (so F(x) = (f_1(x), ..., f_q(x)) for x \in D), then F is differentiable at a iff each f_i is differentiable at a. In this case, we have dF(a)(h) = \sum_{i=1}^{q} \frac{d}{dx_i} f_i(a)(h) \text{ for all } h, i = 1, ..., q.

Proof: Note that f_i(x) = \frac{d}{dx_i} F(x) for each i and x \in D. Now suppose that F is differentiable at a and let i = 1, ..., q. Then

\[ \lim_{h \to 0} \frac{1}{\|h\|} (f_i(a+h) - f_i(a) - f_i(a)(h)) = \lim_{h \to 0} \frac{1}{\|h\|} (\frac{d}{dx_i} F(a+h) - \frac{d}{dx_i} F(a) - \frac{d}{dx_i} F(a)(h)) = 0 \]

Hence, f_i is differentiable at a and dF_i(a)(h) = \frac{d}{dx_i} F(a)(h). Now suppose F is differentiable at a and dF(a) = (df_1(a), ..., df_q(a)). So dF_i(a)(h) = \frac{d}{dx_i} F(a)(h) \text{ for all } h, i = 1, ..., q.

9.2i - Ex: Let f: \mathbb{R}^2 \rightarrow \mathbb{R} be defined by f(x,y) = xy. Show that f is differentiable at (2,3) with dF(2,3)(h_1, h_2) = 3h_1 + 2h_2. (So A = \begin{pmatrix} 3 & 2 \end{pmatrix}.)

We have f(2+h_1, 3+h_2) - f(2,3) - (3h_1 + 2h_2) = (2+h_1)(3+h_2) - 6 - 3h_1 - 2h_2 = h_1 h_2.

So \lim_{h_1, h_2 \to (0,0)} \frac{f(2+h_1, 3+h_2) - f(2,3) - (3h_1 + 2h_2)}{\sqrt{h_1^2 + h_2^2}} = 0. Hence, f is differentiable at (2,3) with dF(2,3)(h_1, h_2) = 3h_1 + 2h_2.
4.2 - continued

Given a linear transform \( L : \mathbb{R}^p \rightarrow \mathbb{R}^q \), let \([L]\) denote the \(q \times p\) matrix \(A\) s.t. \(L = LA\).

Thus, in the notation of Def 9.2c, if \(F\) is diff at \(a\), then \(A = [dF(a)]\).

9.2f - Theorem: Let \(DC\mathbb{R}^p\) and let \(a \in D\). Sps \(F : D \rightarrow \mathbb{R}^q\) is diff at \(a\). Then \(\forall i = 1, \ldots, q \text{ and } j = 1, \ldots, p\), \(\frac{dF_i}{dx_j}(a)\) exists and \([dF(a)] = \left(\frac{dF_i}{dx_j}(a)\right)\).

Proof: Let \([dF(a)] = A = (a_{ij})\). Since \(F\) is diff at \(a\), \(\lim_{h \to 0} \frac{1}{|h|} (F(a+h) - F(a) - Ah) = 0\).

So given \(\forall i = 1, \ldots, q\), \(\lim_{h \to 0} \frac{1}{|h|} \left(\sum_{j=1}^{p} a_{ij} h_j\right) = 0\).

Note, if one knows \(F\) is diff at \(a\), it is easy to find the matrix \([dF(a)] = \left(\frac{dF_i}{dx_j}(a)\right)\).

But the existence of the partials does not guarantee differentiability.

9.2k - Theorem: Let \(DC\mathbb{R}^p\) be open, \(a \in D\) & let \(f : D \rightarrow \mathbb{R}^q\) be a transf. Sps that \(\forall i = 1, \ldots, q\), \(\forall j = 1, \ldots, p\), \(\frac{dF_i}{dx_j}\) exists on \(D\) and is cont at \(a\). Then \(F\) is diff at \(a\).

In particular, if \(F\) is \(C^1\) on \(U\), then \(F\) is diff at every pt. in \(D\).

Proof: By Prop. 9.2 h it suffices to prove this in the case \(q = 1\). We prove the result when \(p = 2\), then the general case is similar but messier.

Let \(\exists \delta > 0\). We must show that \(\exists \delta > 0\). Sps \((a_1, a_2) \in \mathbb{R}^2\):

\[0 < ||(h_1, h_2)|| < \delta\]

Then \(\forall \epsilon > 0\), \(f(a_1, a_2) + \epsilon ((h_1, h_2)) \in \mathbb{R}^2\).

By the ct of \(\frac{df}{dx_1}, \frac{df}{dx_2}\) at \((a_1, a_2)\), \(\exists \delta > 0\) s.t. \(B_\delta((a_1, a_2)) \subseteq D\) and \(\forall \epsilon > 0\), \(\exists \delta > 0\).

Now let \((h_1, h_2) \in \mathbb{R}^2\). Then \((a_1 + h_1, a_2 + h_2) \in D\). By the MVT applied twice, \(\exists \xi, \eta \in (0, 1)\) s.t. \(f(a_1 + \xi h_1, a_2 + \eta h_2) - f(a_1, a_2) = \frac{df}{dx_1}(a_1 + \xi h_1, a_2 + \eta h_2) h_1 \epsilon + \frac{df}{dx_2}(a_1 + \xi h_1, a_2 + \eta h_2) h_2 \epsilon\).

Then we have

\[|\frac{df}{dx_1}(a_1 + \xi h_1, a_2 + \eta h_2) - \frac{df}{dx_1}(a_1, a_2)| \leq \frac{df}{dx_1}(a_1 + \xi h_1, a_2 + \eta h_2) h_1 |\xi| + |\frac{df}{dx_2}(a_1 + \xi h_1, a_2 + \eta h_2) - \frac{df}{dx_2}(a_1, a_2)| h_2 |\eta| \leq \epsilon \|(h_1, h_2)\|\]

Thus \((*)\) holds and so \(f\) is diff at \((a_1, a_2)\).

9.2l - Note: Sps \(F : D \rightarrow \mathbb{R}^q\) is \(C^1\), then \(F\) is diff on \(D\) and \([dF(a)] = \left(\frac{dF_i}{dx_j}(a)\right)\). This matrix is called the Jacobian matrix in many books.

9.2m - Ex: Let \(F : \mathbb{R}^2 \rightarrow \mathbb{R}^2\) be defined by \(F(x, y) = (x^2 - y^2, 2xy)\). Then \(F\) is \(C^1\) on \(\mathbb{R}^2\) and \([dF(x, y)] = \left(\begin{array}{cc} 2x & -2y \\ 2y & 2x \end{array}\right)\).

9.2m - Ex: Let \(f : \mathbb{R}^2 \rightarrow \mathbb{R}\) be given by \(f(x, y) = \frac{xy}{x^2 + y^2}\) if \((x, y) \neq (0, 0)

We have \(\frac{2f}{dx}(0, 0) = 0 = \frac{2f}{dy}(0, 0)\), but since \(f\) is not cont at \((0, 0)\), it is not diff at \((0, 0)\) because it is not cont at \((0, 0)\).