9.7 The Implicit Function Theorem

Consider the curve \( z^2 + y^2 = 1 \) or equivalent \( f(x,y) = x^2 + y^2 - 1 = 0 \). We would like to regard \( y \) as an implicitly defined function of \( x \) at least locally. This may not work at pts where \( \frac{df}{dy}(x,y) = 0 \), but otherwise we can locally solve for \( y \) as a function of \( x \) near a pt \((a,b)\), where \( f(a,b) = 0 \).

9.7a - Implicit Function Theorem - Basic Case

Let \( U \subseteq \mathbb{R}^2 \) be open, let \( f : U \to \mathbb{R} \) be smooth and let \((a,b) \in U \). Say that \( f(a,b) = 0 \) and \( \frac{df}{dy}(a,b) \neq 0 \). Then \( f \) has a smooth local inverse; hence:

- By the Inverse Function Theorem 9.6e there is a smooth local inverse \( g \) of \( f \) near \((a,b) \).

9.7b - Notation for the General Case:

Let \( p, q \geq 1 \), let \( U \subseteq \mathbb{R}^p \times \mathbb{R}^q \) be open and let \( f : U \to \mathbb{R}^q \) be smooth. We write \((x,y) \in \mathbb{R}^p \times \mathbb{R}^q \) as \((x_1, \ldots, x_p, y_1, \ldots, y_q)\) and we write

\[
[Df(x,y)] = \begin{bmatrix}
\frac{df_1}{dx_1} & \cdots & \frac{df_1}{dx_p} \\
\vdots & \ddots & \vdots \\
\frac{df_q}{dx_1} & \cdots & \frac{df_q}{dx_p}
\end{bmatrix}
\]

and similarly

\[
[Df(x_1, \ldots, x_p, y_1, \ldots, y_q)] = \begin{bmatrix}
\frac{df_1}{dx_1} & \cdots & \frac{df_1}{dx_p} \\
\vdots & \ddots & \vdots \\
\frac{df_q}{dx_1} & \cdots & \frac{df_q}{dx_p}
\end{bmatrix}
\]

9.7c - Implicit Function Theorem - General Case:

With \( U', f \) as above, let \((a,b) \in U \) st. \( f(a,b) = 0 \). Then \( f \) is smooth on \( U \) and \( f(a,b) = 0 \) and \( \frac{df}{dy}(a,b) \neq 0 \). There is an open set \( V \subseteq \mathbb{R}^q \) and an open \( W \subseteq \mathbb{R}^p \) st. \( f \) is smooth on \( W \) and \( f(a,b) = 0 \).

9.7c - Ex: Take \( p = 2, q = 1 \). Let \( f(x,y) = x^2 + y^2 - 9 \) and take \((a,b) = (2,2,1)\). We have

\[
[Df(x,y)] = \begin{bmatrix}
2x & 2y \\
0 & 2y
\end{bmatrix}
\]

so \( \frac{df}{dy}(2,2,1) = 2 \neq 0 \); hence \( f \) is smooth on \( \mathbb{R}^2 \) and \( Df(2,2,1) \) is invertible.

9.7d - Ex: Take \( p = 1, q = 2 \). Let \( f(x,y) = x^2 + 2xy + 2y^2 - 8 \) and take \((a,b) = (2,2,1)\). Then \( f \) is smooth on \( \mathbb{R}^2 \) and

\[
[Df(x,y)] = \begin{bmatrix}
2x & 2y \\
2y & 4y
\end{bmatrix}
\]

so \( \frac{df}{dy}(2,2,1) = 4 \neq 0 \); hence \( f \) is smooth on \( \mathbb{R}^2 \) and \( Df(2,2,1) \) is invertible.

Thus \( g(2) = (2,3) \) is invertible.