1. In each case determine whether the given series is absolutely convergent, conditionally convergent or divergent. Prove your answer, be sure to cite any tests you use by name whenever possible.
   a. \( \sum_{n=3}^{\infty} (-1)^n \frac{\sqrt{\ln n}}{n} \)
   b. \( \sum_{n=1}^{\infty} \frac{(-10)^n n^{10}}{n!} \)
   c. \( \sum_{n=1}^{\infty} \frac{n}{((-1)^n + 3)^n} \)
   d. \( \sum_{n=2}^{\infty} ne^{-n} \sin n \)
   e. \( \sum_{n=3}^{\infty} \frac{\sqrt{n} + 5}{2n^2 - 3n - 5} \)
   f. \( \sum_{n=1}^{\infty} (-1)^n \cos \left( \frac{1}{n} \right) \)

2. Let \( \{a_n\}_n \) and \( \{b_n\}_n \) be two positive sequences. Suppose that \( \sum_{n=1}^{\infty} a_n \) diverges and \( a_n/b_n \to 0 \). Prove that \( \sum_{n=1}^{\infty} b_n \) diverges.

3. Let \( \{a_n\}_n \) be a positive sequence and suppose that \( \sum_{n=1}^{\infty} a_n \) converges. Prove that \( \sum_{n=1}^{\infty} a_n^r \) converges for all \( r \geq 1 \). Show that this need not be the case for \( r \in (0, 1) \).

4. Consider the infinite series of functions given below.
   \[ \sum_{n=1}^{\infty} \frac{n^2 \cos nx}{3^n} \]
   a. Prove that the infinite series converges uniformly on \( \mathbb{R} \) to a continuous function \( f \).
   b. Prove that \( f \) is differentiable on \( \mathbb{R} \) and find its derivative.
   c. Show that \( f \) is integrable on \( [0, \pi/2] \) and evaluate \( \int_0^{\pi/2} f(x) \, dx \).

5. Consider the following power series. Find its radius of convergence and the set of points \( x \) at which the series converges.
   \[ \sum_{n=0}^{\infty} \frac{(-1)^n(x + 7)^n}{4^n(2n + 3)} \]

6. Let \( a \in \mathbb{R} \) and let \( f \) be defined by \( f(x) = x^a \). Prove that \( f \) is infinitely differentiable at \( a \) and find its Taylor series at \( a \) and show that it converges to \( f \) on \( \mathbb{R} \).

7. Find a power series representation at \( a = 0 \) for the function \( g(x) = \frac{x^2}{1+x^2} \) and find its radius of convergence.

8. Use the definition to find the Taylor series for the function \( h(x) := \ln x \) at \( a = 1 \). Prove that the Taylor series converges to \( h \) on the interval \( I := (0, 2) \). Show that convergence is uniform on any closed subinterval of \( I \).

9. Use Taylor’s formula to prove that the Taylor series for \( f(x) = \sin 2x \) at \( a = \pi \) converges to \( f \) on \( \mathbb{R} \).

10. Suppose that \( x, y \in \mathbb{R}^d \) are orthogonal. Prove that \( \|2x - y\|^2 = 4\|x\|^2 + \|y\|^2 \).

11. Prove the Parallelogram Law: For all \( x, y \in \mathbb{R}^d \),
   \[ \|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2 \]

12. Consider the sequence \( \{x_n\}_n \) in \( \mathbb{R}^3 \) given by
   \[ x_n := (e^{-n}, \cos(2n\pi/5), \sin(2n\pi/5)) \]
   Find five convergent subsequences that converge to different limits and find these limits.
13. Prove that the map \( x \in \mathbb{R}^d \mapsto \|x\|_1 \) defined below is a norm on \( \mathbb{R}^d \).
\[
\|x\|_1 = |x_1| + \cdots + |x_d|.
\]

   Prove that \( \|x\| \leq \|x\|_1 \leq d\|x\| \) for all \( x \in \mathbb{R}^d \).

14. Prove that the following subset of \( \mathbb{R}^2 \) is open:
\[
U = \{(x, y) : |x| < 1, |y| < 1\}.
\]

15. A set \( X \subset \mathbb{R}^d \) is said to be bounded if there is a positive number \( M \) such that \( \|x\| \leq M \) for all \( x \in X \). Prove that \( X \) is bounded iff there is a positive number \( L \) such that \( |x_i| \leq L \) for all \( x \in X \) and \( i = 1, \ldots, d \).

16. Let \( E \subset \mathbb{R}^d \). Prove that \((E)^c = (E^c)^c\). (Hint: Use HW 5.2(a).)

17. Let \( E \subset \mathbb{R}^d \). Let \( x \in E \); prove that \( x \in \partial E \) if and only if for any \( r > 0 \), \( E \cap B_r(x) \neq \emptyset \) and \( E^c \cap B_r(x) \neq \emptyset \). (Hint: Use HW 5.2(b).)

18. Let \( U \subset \mathbb{R}^d \) be an open subset. Either prove that \( U = (\overline{U})^c \) or find a counterexample.

19. Let \( A, B \subset \mathbb{R}^d \). Prove that \( \overline{A \cup B} = \overline{A} \cup \overline{B} \) but \( A \cap B \neq \overline{A} \cap \overline{B} \) in general. Show that \( \overline{A \cap B} \subset \overline{A} \cap \overline{B} \).

20. Let \( E \subset \mathbb{R}^d \). Prove that \( \partial E = E \cap \overline{E^c} \).

21. Let \( K \subset \mathbb{R}^d \) be a compact set. Prove that every sequence in \( K \) has a convergent subsequence which converges to a point in \( K \).

22. Let \( \{x_n\}_n \) be a convergent sequence in \( \mathbb{R}^d \) with limit \( x_\infty \). Prove that the following set is compact.
\[
K := \{x_\infty\} \cup \{x_n : n \in \mathbb{N}\}.
\]

   Show by example that \( \{x_n : n \in \mathbb{N}\}_n \) need not be compact.

23. Let \( \{K_n\}_n \) be a sequence of compact subsets of \( \mathbb{R}^d \) and suppose that \( K_1 \cap \cdots \cap K_n \neq \emptyset \). Prove that \( \bigcap_n K_n \neq \emptyset \).

24. For \( X \subset \mathbb{R}^d \), define \( \text{diam } X = \sup \{\|x - y\| : x, y \in X\} \). Let \( \{K_n\}_n \) be a sequence of nonempty compact subsets of \( \mathbb{R}^d \) such that \( \text{diam } K_n \to 0 \) and \( K_{n+1} \subset K_n \) for all \( n \in \mathbb{N} \).

   a. Prove that there is a point \( x_0 \in \mathbb{R}^d \) such that \( \bigcap_n K_n = \{x_0\} \).

   b. Let \( \{x_n\}_n \) be a sequence in \( \mathbb{R}^d \) such that \( x_n \in K_n \) for all \( n \in \mathbb{N} \). Prove that \( \{x_n\}_n \) is Cauchy and that \( x_n \to x_0 \).

25. Consider the following set \( E \). Find \( E^c \), \( \overline{E} \) and \( \partial E \). Is \( E \) is bounded, open, closed or compact?
\[
E = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 4\} \cap \{(x, y) \in \mathbb{R}^2 : xy < 0\}.
\]

26. Let \( y_0 \in \mathbb{R}^d \) and let \( K \subset \mathbb{R}^d \) be compact. Prove that there is \( x_0 \in K \) such that
\[
\|x_0 - y_0\| = \inf_{x \in K} \|x - y_0\|.
\]

27. Let \( K_1, K_2 \subset \mathbb{R}^d \) be two compact subsets such that \( K_1 \cap K_2 = \emptyset \). Prove that there are two open subsets \( U_1, U_2 \subset \mathbb{R}^d \) such that \( K_1 \subset U_1 \), \( K_2 \subset U_2 \) and \( U_1 \cap U_2 = \emptyset \).

28. If possible, give an example of an open cover \( \mathcal{U} \) of \( B_1(0) \) which does not have a finite subcover. Otherwise, prove that it is impossible.