Test II will cover sections 8.1-8.3, 8.5, 9.1-9.3, 9.5. You may bring a formula sheet to the test (one side only); please do not include sample proofs. Review the sections covered and make sure that you know the key theorems and definitions and how to apply them. Go over the solutions to the quizzes and the homework. In addition, here is a list of sample questions.

There will be a supplementary review session Tuesday afternoon 3:15-4:45 in DMS 315. This review sheet may be updated.

1. Let \( f : \mathbb{R}^2 \setminus \{(0,0)\} \to \mathbb{R} \) be given below. Show that the limit of \( f \) along any line through the origin is 0. Show that the limit does not exist at \((0,0)\).

\[
f(x, y) := \frac{x^2 y}{x^4 + y^2}
\]

2. Let \( f : \mathbb{R}^2 \to \mathbb{R} \) be given below. Show that \( f \) is not continuous at \((0,0)\).

\[
f(x, y) := \left\{
\begin{array}{ll}
x^2 + y^2 & \text{if } (x, y) \neq (0,0), \\
1 & \text{if } (x, y) = (0,0).
\end{array}
\right.
\]

3. Let \( F : \mathbb{R}^p \to \mathbb{R}^q \) be a continuous function. Either prove that \( F(U) \) is open for any open set \( U \subset \mathbb{R}^p \) or find a counterexample.

4. Let \( v \in \mathbb{R}^d \) and let \( L : \mathbb{R}^d \to \mathbb{R} \) be the linear map defined by \( L(x) = v \cdot x \). Show that \( L \) is linear and that \( ||[L]||_M = ||v|| \).

5. Show that the transformation \( T : \mathbb{R}^2 \to \mathbb{R}^2 \) which reflects a point about the line \( x + y = 1 \) is affine but not linear.

6. Let \( D := [-\pi, \pi] \times [-\pi, \pi] \) and define \( F : D \to \mathbb{R}^3 \) by

\[
F(u, v) = (\cos u + 2 \cos v, \cos u + 2 \sin v, \sin u).
\]

Explain why \( F \) is continuous and describe its range. Show that \( F \) differentiable at \((0,0)\) and find \([dF(0, 0)]\), its differential matrix at \((0,0)\).

7. Let \( f : \mathbb{R}^2 \to \mathbb{R} \) be given below. Determine whether \( f \) is continuous at \((0,0)\). Justify your answer.

\[
f(x, y) := \left\{
\begin{array}{ll}
x^3 y^2 & \text{if } (x, y) \neq (0,0), \\
0 & \text{if } (x, y) = (0,0).
\end{array}
\right.
\]

8. Show that the transformation \( L \) from \( \mathbb{R}^3 \) to \( \mathbb{R}^3 \) which projects each point onto the plane consisting of all points \((x,y,z)\) which satisfy the equation \( x + 2y + 3z = 0 \) is linear (and find the associated matrix). Show that the transformation \( T \) which projects each point onto the plane which consists of points \((x,y,z)\) satisfying the equation \( x + 2y + 3z = 6 \) is affine but not linear.

9. Find parametric equations for the plane \( P \) consisting of all points \((x,y,z)\) which satisfy the equation \( x + 2y + 3z = 6 \). Find an affine transformation \( S : \mathbb{R}^2 \to \mathbb{R}^3 \) such that the range of \( S \) is \( P \).

10. Let \( f : \mathbb{R}^d \to \mathbb{R} \) be a continuous function such that for every \( \varepsilon > 0 \) there is an \( M > 0 \) such that \( |f(x)| < \varepsilon \) for all \( x \in \mathbb{R}^d \) such that \( ||x|| > M \). Prove that \( f \) is uniformly continuous on \( \mathbb{R}^2 \).

11. Let \( A \) be a \( d \times d \) matrix and suppose there is an orthonormal basis of \( \mathbb{R}^d \) consisting of eigenvectors of \( A \), \( v_1, \ldots, v_d \), with associated eigenvalues \( \lambda_1, \ldots, \lambda_d \). Prove that \( ||A||_M = \max\{\lambda_i : 1 \leq i \leq d\} \).

12. Let \( A \) be an \( r \times q \) matrix and let \( B \) be a \( q \times p \) matrix. Prove that \( ||AB||_M \leq ||A||_M ||B||_M \).

13. Let \( f : \mathbb{R}^2 \to \mathbb{R} \) be defined by

\[
f(x, y) := \left\{
\begin{array}{ll}
x^2 y & \text{if } (x, y) \neq (0,0), \\
0 & \text{if } (x, y) = (0,0).
\end{array}
\right.
\]

Prove that \( \frac{\partial f}{\partial x} \) exists on \( \mathbb{R}^2 \) but that it is not continuous at \((0,0)\). Is \( f \) differentiable at \((0,0)\)?

14. Let \( f : \mathbb{R}^2 \to \mathbb{R} \) be defined by \( f(x, y) = (2x - xy, 3y + x^2) \). Use the definition to prove that \( f \) is differentiable at \((1, -2)\) and find the differential \( dF(1, -2) \). Find the affine transformation \( T \) that best approximates \( F \) near \((1, -2)\). Is \( F \) a \( C^1 \) transformation?

15. Let \( f : \mathbb{R} \to \mathbb{R} \) be defined by

\[
f(x) := \left\{
\begin{array}{ll}
x^2 \sin(1/x) & \text{if } x \neq 0,
0 & \text{if } x = 0.
\end{array}
\right.
\]

Prove that \( f \) is differentiable on \( \mathbb{R} \) but \( f' \) is not continuous at \( 0 \).

16. Find, if possible, an example of a function \( f : \mathbb{R}^2 \to \mathbb{R} \) such that \( f \) is differentiable at \((0,0)\), \( \frac{\partial f}{\partial x} \) exists on \( \mathbb{R}^2 \) but \( \frac{\partial f}{\partial x} \) is not continuous at \((0,0)\).
17. Let \( I, J \) be open subsets of \( \mathbb{R} \), let \( a \in I \), let \( b \in J \) and let \( f : I \to \mathbb{R} \) and \( g : J \to \mathbb{R} \) be functions. Suppose that \( f \) is differentiable at \( a \) and \( g \) is differentiable at \( b \).
   a. Prove that the transformation \( F_1 : I \times J \to \mathbb{R}^2 \) given by \( F_1(x, y) := (f(x), g(y)) \) for \((x, y) \in I \times J \) is differentiable at \((a, b)\) and find its differential matrix \([dF_1(a, b)]\).
   b. Use the Chain Rule to prove that the transformation \( F_2 : I \times J \to \mathbb{R}^2 \) given by \( F_2(x, y) := (f(x) + g(y), f(x)g(y)) \) for \((x, y) \in I \times J \) is differentiable at \((a, b)\) and find its differential matrix \([dF_2(a, b)]\).
18. Let \( U \) be a neighborhood of \( V \) in \( \mathbb{R}^d \). Suppose that \( G : U \to \mathbb{R}^q \) and \( F : V \to \mathbb{R}^r \) are smooth transformations such that \( G(U) \cap V \neq \emptyset \). Prove that \( G \circ F \) is smooth on \( W := G^{-1}(V) \).
19. Let \( A \) and \( B \) be distinct points in \( \mathbb{R}^d \). Show that for all \( x \in \mathbb{R}^d \), \( x \) is differentiable at \( a \) if and only if \( x \) is differentiable at \( b \).
20. Let \( U \subset \mathbb{R}^p \) and \( V \subset \mathbb{R}^q \) be open sets. Suppose that \( G : U \to \mathbb{R}^q \) and \( F : V \to \mathbb{R}^r \) are smooth transformations such that \( G(U) \cap V \neq \emptyset \). Prove that \( G \circ F \) is smooth on \( W := G^{-1}(V) \).
21. Prove that the composition of affine transformations is an affine transformation.
22. Let \( U \subset \mathbb{R}^d \), let \( a \in U \) and let \( x \in \mathbb{R}^d \). Suppose that \( f : U \to \mathbb{R} \) is differentiable at \( a \) and define the function \( g \) on a neighborhood \( V \) of \( 0 \) by \( g(t) := f(a + tx) \) for \( t \in V \). Prove that \( g \) is differentiable at \( 0 \) and \( g'(0) = df(a)(x) \).
23. Let \( U := \{(x, y) : x > 0\} \subset \mathbb{R}^2 \) and let \( F : U \to \mathbb{R}^2 \) be a smooth transformation. Prove that the transformation \( G \) defined on \( V := \{(s, t) : |s| < \pi/2, t > 0\} \) by \( G(s, t) := F(s \cos t, s \sin t) \) is differentiable at \((\sqrt{2}, \pi/4)\) and find the differential matrix \([dG(\sqrt{2}, \pi/4)]\).
24. Let \( A \) be a \( d \times d \) symmetric matrix (so \( x \cdot Ay = y \cdot Ax \)) and let \( f : \mathbb{R}^d \to \mathbb{R} \) be defined by \( f(x) := x \cdot Ax \) for all \( x \in \mathbb{R}^d \). Prove that for all \( a \in \mathbb{R}^d \), \( f \) is differentiable at \( a \) and that its differential at \( a \) is given by \( df(a)(x) = 2a \cdot Ax \) for \( x \in \mathbb{R}^d \).
25. Let \( a \in \mathbb{R}^d \) and let \( r > 0 \). Prove that \( B_r(a) \) is convex.
26. Let \( f : \mathbb{R}^2 \to \mathbb{R} \) be defined by \( f(x, y) := xy(1 - x^2 - y^2) \).
   a. Find all critical points of \( f \), that is, all \( a \in \mathbb{R}^2 \) such that \( df(a) = 0 \).
   b. Find each point \( a \in \mathbb{R}^2 \) at which \( f \) has a local maximum, a local minimum or a saddle point.
   c. Does \( f \) have an absolute maximum or minimum on \( \mathbb{R}^2 \)?
   d. Let \( D := \{(x, y) : 0 \leq x \leq 2, -x \leq y \leq x\} \) and find
      \[
      \sup_{(x,y) \in D} f(x, y) \quad \text{and} \quad \inf_{(x,y) \in D} f(x, y).
      \]
27. Let \( U \subset \mathbb{R}^p \) be an open set, let \( F : U \to \mathbb{R}^q \) be a differentiable transformation and let \( a, b \in U \) be distinct points such that \([a, b] \subset U \). Either prove that there is a point \( c \in [a, b] \) such that
      \[
      F(b) - F(a) = dF(c)(b - a)
      \]
      or find an example of a transformation \( F \), an open set \( U \subset \mathbb{R}^p \) and distinct points \( a, b \in U \) where there is no such \( c \).
28. Let \( f : \mathbb{R}^3 \to \mathbb{R} \) be defined by \( f(x, y, z) := xy + yz + xz \). Find all critical points of \( f \) and prove that \( f \) has no local extrema. Show that the function \( g : \mathbb{R}^3 \to \mathbb{R} \) given by \( g(x, y, z) := f(x, y, z) + x^2 + y^2 + z^2 \) has a local minimum.