Please attempt exactly four of the following problems. Clearly indicate the one you are omitting. Each problem is worth 10 points. For full credit please show all work and use complete English sentences. Cite any theorem you use by name, if at all possible, and make sure to check its hypotheses. No wireless devices are permitted.

1. In each case determine whether the given series is absolutely convergent, conditionally convergent or divergent. Prove your answer and be sure to cite any tests you use by name whenever possible.

   a. \[ \sum_{n=1}^{\infty} (-1)^n \frac{4}{3n+7} \]

   We apply the AST with \( a_n = \frac{4}{3n+7} \). Note that for all \( n \), \( a_n > 0 \) &
   \[ a_{n+1} = \frac{4}{3(n+1)+7} = \frac{4}{3n+10} \leq \frac{4}{3n+7} = a_n. \]
   Moreover, \( a_n \to 0 \).

   So AST applies and we conclude that \( \sum_{n=1}^{\infty} (-1)^n \frac{4}{3n+7} \) converges.

   We claim that \( \sum_{n=1}^{\infty} \frac{4}{3n+7} \) diverges. We set \( b_n = \frac{1}{n} \) and apply the LCT.

   Since \( \frac{a_n}{b_n} = \frac{4}{3n+7}, \frac{1}{n} = \frac{4}{3n+7/n} \to \frac{4}{3} > 0 \) \& \( \sum_{n=1}^{\infty} \frac{1}{n} \)

   diverges, we conclude that \( \sum_{n=1}^{\infty} \frac{4}{3n+7} \) diverges also. So the given series is not AC and it is thus CC.

   b. \[ \sum_{n=1}^{\infty} \frac{ne^n}{n!} \]

   We apply the Ratio Test:
   \[ \left| \frac{a_{n+1}}{a_n} \right| = \frac{(n+1)e^{n+1}}{(n+1)!} \cdot \frac{n!}{ne^n} = \frac{e}{n} \to 0 < 1. \]

   Hence, by the Ratio Test the series is AC.
2. Let \( f(x) = e^{-x} \) for \( x \in \mathbb{R} \).
   a. Find the Taylor series for \( f \) at \( a = 0 \).

   We have \( e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \) for all \( x \in \mathbb{R} \).

   Hence \( e^{-x} = \sum_{n=0}^{\infty} \frac{(-x)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!} \) for all \( x \in \mathbb{R} \).

   Thus \( \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!} \) is the Taylor series for \( f \) at \( a = 0 \).

   b. Use Taylor’s formula to prove that the Taylor series for \( f \) converges to \( f \) on \( \mathbb{R} \).

   Note \( f^{(n)}(x) = (-1)^n e^{-x} \) for all \( x \in \mathbb{R} \) and \( n \geq 0 \).

   So \( R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!} x^{n+1} = \frac{(-1)^n e^{-c} x^{n+1}}{(n+1)!} \) for some \( c \) between 0 and \( x \).

   Since \( |f^{(n+1)}(c)| = e^{-c} \leq e^{1x} \) for all \( c \) between 0 and \( x \), we have

   \[
   |R_n(x)| = \frac{e^{1x} |x|^{n+1}}{(n+1)!} \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty.
   \]

   Note: \( \frac{r^n}{n!} \rightarrow 0 \) for all \( r \) since \( \sum_{n=0}^{\infty} \frac{r^n}{n!} \) converges.

   Therefore \( T_n(x) \rightarrow e^{-x} \) for all \( x \in \mathbb{R} \).
3. Consider the infinite series of functions given below.
\[
\sum_{n=1}^{\infty} \frac{\cos nx}{2^n}
\]

a. Prove that the infinite series converges uniformly on \(\mathbb{R}\) to a continuous function \(f\).

We use the Weierstrass M-test to show that the series converges uniformly on \(\mathbb{R}\). Let \(M_n = \frac{1}{2^n}\). Then since \(|\cos nx| \leq 1\) for all \(n \in \mathbb{N}, x \in \mathbb{R}\), we have for each \(n \in \mathbb{N}\):
\[
\left| \frac{\cos nx}{2^n} \right| \leq \frac{1}{2^n} = M_n
\]
for all \(x \in \mathbb{R}\).

Moreover, \(\sum_{n=1}^{\infty} M_n = \sum_{n=1}^{\infty} \frac{1}{2^n} = 1\), so \(\sum_{n=1}^{\infty} M_n\) converges. Hence by WMT the series converges uniformly (and absolutely) to a function \(f\) on \(\mathbb{R}\).

Since \(g_n(x) = \sum_{k=1}^{n} \frac{\cos nx}{2^k}\) is cts. and \(g_n \to f\) uniform on \(\mathbb{R}\),
\(f\) must also be cts.

b. Find an explicit expression for \(F\) as an infinite series of functions where
\[
F(x) = \int_0^x f(t) \, dt \quad \text{for } x \in \mathbb{R}.
\]

Since \(f\) is cts. it is integrable on any closed interval \([a,b] \subseteq \mathbb{R}\).

Moreover, since convergence is uniform we may integrate termwise.

Hence for all \(x \in \mathbb{R}\) we have
\[
F(x) = \int_0^x f(t) \, dt = \int_0^x \sum_{n=1}^{\infty} \frac{\cos nx}{2^n} \, dt = \sum_{n=1}^{\infty} \frac{1}{2^n} \int_0^x \cos nx \, dt
\]
\[
= \sum_{n=1}^{\infty} \frac{1}{2^n} \cdot \frac{1}{n} \left( \sin nx - \sin 0 \right)
\]
\[
= \sum_{n=1}^{\infty} \frac{\sin nx}{n2^n}
\]
4. With \( E \subset \mathbb{R}^2 \) as given below, find \( E^o \), \( \overline{E} \) and \( \partial E \). Determine if \( E \) is bounded, open or closed.

\[
E = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\} \cap \{(x, y) \in \mathbb{R}^2 : x \geq 0\}.
\]

The interior \( E^o \) consists of all pts in \( E \) except pts on the y-axis, \( x = 0 \). So

\[
E^o = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\} \cap \{(x, y) \in \mathbb{R}^2 : x > 0\}.
\]

The closure \( \overline{E} \) consists of all pts in \( E \) as well as all pts on the boundary. So

\[
\overline{E} = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\} \cup \{(x, y) \in \mathbb{R}^2 : x \geq 0\}.
\]

The boundary is given by \( \partial E = \overline{E} \setminus E^o \). So

\[
\partial E = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1, x > 0\} \cup \{(x, y) \in \mathbb{R}^2 : x = 0, -1 \leq y \leq 1\}.
\]

Since \( E \subset \overline{E}(0,0) \), \( E \) is bounded. ( \( \forall (x, y) \in E, \| (x, y) \| \leq 1 \)

\( E \) is not open because \( E \neq E^o \) and \( E \) is not closed because \( E \notin \overline{E} \).
5. Let $K \subset \mathbb{R}^d$ be a compact set and let $\mathcal{U} = \{U_t : t > 0\}$ be an open cover of $K$ such that $U_s \subset U_t$ if $s < t$. Prove that $K \subset U_t$ for some $t > 0$.

Since $K$ is compact and $\mathcal{U} = \{U_t : t > 0\}$ is an open cover of $K$, there is a finite subcover, that is, $\exists t_1, \ldots, t_n > 0$ s.t. $K \subset U_{t_1} \cup U_{t_2} \cup \cdots \cup U_{t_n}$. By relabeling if necessary we may assume $t_1 < t_2 < \cdots < t_n$. Then $U_{t_i} \subset U_{t_n}$ for all $i = 1, \ldots, n$. Hence, $K \subset U_{t_1} \cup U_{t_2} \cup \cdots \cup U_{t_n} = U_{t_n}$. 