Please attempt exactly five of the following six problems. Clearly indicate the one you are omitting. Each problem is worth 10 points. For full credit please show all work and use complete English sentences. Cite any theorem you use by name, if possible, and make sure to check its hypotheses. No wireless devices are permitted.

1. Let \( D := \{(x, y, z) \in \mathbb{R}^3 : |x| \leq 1, |y| \leq 1 \text{ and } |z| \leq 1\} \) and let \( f : D \rightarrow \mathbb{R} \) be defined by \( f(x, y, z) := xy + xz + yz \) for all \((x, y, z) \in D\). Determine whether \( f \) is uniformly continuous.

Since \( D \) is closed and bounded, it is compact by the Heine-Borel theorem. The function \( f(x, y, z) = xy + xz + yz \) is a polynomial and therefore continuous.

A continuous function defined on a compact set must be uniformly continuous. Therefore, \( f \) is uniformly continuous.
2. Let \( a = (a_1, \ldots, a_n) \in \mathbb{R}^n \) be nonzero and let \( A = (a_1 \ldots a_n) \) be the \( 1 \times n \) matrix with entries \( a_1, \ldots, a_n \) (so that \( Ax = a \cdot x \) for \( x \in \mathbb{R}^n \)). Prove that \( \|A\|_M = \|a\| \).

Recall that \( \|A\|_M = \text{Sup} \frac{1}{2} \|Ax\| : \|x\| \leq 1/2 \).

We first show \( \|a\| \) is an upper bound for \( \frac{1}{2} \|Ax\| : \|x\| \leq 1/2 \).

Let \( x \in \mathbb{R}^n \) with \( \|x\| \leq 1 \). Then by the Cauchy-Schwarz inequality we have
\[
\|Ax\| = |a \cdot x| \leq \|a\| \|x\| \leq \|a\| \quad \text{(since } \|x\| \leq 1 \text{)}.
\]
Hence \( \|a\| \) is an upper bound for \( S = \frac{1}{2} \|Ax\| : \|x\| \leq 1/2 \).

Thus \( \|A\|_M \leq \|a\| \).

For the reverse inequality, set \( x = \frac{1}{\|a\|} a \); then \( \|x\| = 1 \)
and so \( \|a\| = |a \cdot a| = \|Ax\| \in S \).

Hence, \( \|a\| \leq \|A\|_M \).

Thus \( \|A\|_M = \|a\| \).
3. Let \( f : \mathbb{R}^2 \to \mathbb{R} \) be given below. Determine whether \( f \) is continuous at \((0,0)\). Justify your answer.

\[
f(x, y) := \begin{cases} 
\frac{x^2 y^2}{(x^2 + y^2)^2} & \text{if } (x, y) \neq (0, 0), \\
0 & \text{if } (x, y) = (0, 0).
\end{cases}
\]

We prove that \( f \) is not continuous at \((0,0)\).

Let \( (z_n) \) be the sequence in \( \mathbb{R}^2 \) given by \( z_n = (\frac{1}{n}, \frac{1}{n}) \).

Then \( f(z_n) = \frac{(\frac{1}{n})^2 (\frac{1}{n})^2}{((\frac{1}{n})^2 + (\frac{1}{n})^2)^2} = \frac{1}{4} \) and hence \( \lim_{n \to \infty} f(z_n) = \frac{1}{4} \).

Note that \( z_n \to 0 \), but \( \lim_{n \to \infty} f(z_n) = \frac{1}{4} \neq 0 = f(0,0) \).

Hence, by the sequential characterization of continuity, \( f \) is not continuous at \((0,0)\).
4. Let $F : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $F(x, y) = (y^2 - 3x, x^2 + 5y)$. Use the definition to prove that $F$ is differentiable at $(1, 1)$ and find $[dF(1, 1)]$ the differential matrix at $(1, 1)$.

Set $A = \begin{pmatrix} -3 & 2 \\ 2 & 5 \end{pmatrix}$ and $a = (1, 1)$; we prove

(*) $\lim_{h \to 0} \frac{1}{\|h\|} \left( F(a+h) - F(a) - Ah \right) = O$.

For $h = (h_1, h_2)$ we have $F(a) = (-2, 6)$ and

$F(a+h) - F(a) - Ah$

$= \left( (1 + 2h_2 - 3h_1 + 2 + 3h_1 - 2h_2, 1 + 2h_1 + h_2^2 + 5 + 5h_2 - 6 - 2h_1 - 5h_2) \right)$

$= (h_1^2, h_2^2)$

Since $|h_1|, |h_2| \leq \|h\|$ we have by the triangle inequality

$\| (h_1^2, h_2^2) \| \leq 2 \| h \|$. \hspace{1cm} \text{(1)}$

Hence $\| F(a+h) - F(a) - Ah \| = \| (h_1^2, h_2^2) \| \leq 2 \| h \|$. \hspace{1cm} \text{(2)}$

Since $\lim_{h \to 0} 2 \| h \| = 0$, we have by the Squeeze theorem that

$\lim_{h \to 0} \frac{1}{\|h\|} \| F(a+h) - F(a) - Ah \| = 0$.

Hence, (*) holds and $F$ is differentiable at $a = (1, 1)$

and $[dF(1, 1)] = A = \begin{pmatrix} -3 & 2 \\ 2 & 5 \end{pmatrix}$.
5. Let $H : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $H(x, y) = (x^2 \cos y, y \sin x)$. Show that $H$ is a $C^1$ transformation and find $[dH(a, b)]$ its differential matrix at $(a, b)$.

We first check that all first partials of the components of $H$ are continuous:

- $\frac{\partial h_1}{\partial x} = 2x \cos y$, $\frac{\partial h_1}{\partial y} = -x^2 \sin y$
- $\frac{\partial h_2}{\partial x} = y \cos x$, $\frac{\partial h_2}{\partial y} = \sin x$

Since these are all products of elementary functions, they are continuous and therefore $H$ is $C^1$.

Therefore, for all $H$ is differentiable at each pt $(a, b) \in \mathbb{R}^2$ and

$$[dH(a, b)] = \begin{pmatrix}
\frac{\partial h_1}{\partial x}(a, b) & \frac{\partial h_1}{\partial y}(a, b) \\
\frac{\partial h_2}{\partial x}(a, b) & \frac{\partial h_2}{\partial y}(a, b)
\end{pmatrix}$$

$$= \begin{pmatrix}
2a \cos b & -a^2 \sin b \\
b \cos a & \sin a
\end{pmatrix}$$
6. Let $F : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be a smooth transformation and let $G : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be defined by $G(x, y) := F(x^2 - y^2, xy)$ for all $(x, y) \in \mathbb{R}^2$. Prove that $G$ is differentiable at $(2, -1)$ and express $[dG(2, -1)]$, its differential matrix at $(2, -1)$, as a product of matrices.

We define $H : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ by $H(x, y) = (x^2 - y^2, xy)$. Then since all first partials of the component of $H$ are continuous since

$$\frac{\partial h_1}{\partial x} = 2x, \quad \frac{\partial h_1}{\partial y} = -2y,$$
$$\frac{\partial h_2}{\partial x} = y, \quad \frac{\partial h_2}{\partial y} = x,$$

are all continuous, thus $H$ is $C^1$ and thus $H$ is differentiable at $(2, -1)$ and $[dH(2, -1)] = \begin{pmatrix} 4 & 2 \\ -1 & 2 \end{pmatrix}$.

Note that $G = F \circ H$ and that $H(2, -1) = (3, -2)$. Since $F$ is $C^1$, $F$ is differentiable at $(3, -2)$ and hence by the Chain Rule $G = F \circ H$ is differentiable at $(2, -1)$ and

$$[dG(2, -1)] = [dF(H(2, -1))] \cdot [dH(2, -1)]$$
$$= \begin{pmatrix} \frac{\partial f_1}{\partial x}(3, -2) & \frac{\partial f_1}{\partial y}(3, -2) \\ \frac{\partial f_2}{\partial x}(3, -2) & \frac{\partial f_2}{\partial y}(3, -2) \end{pmatrix} \begin{pmatrix} 4 & 2 \\ -1 & 2 \end{pmatrix}$$