Please attempt exactly four of the following five problems. Clearly indicate the one you are omitting. Each of these is worth 10 points. There is also an extra credit problem worth four points on the last page. For full credit please show all work and use complete English sentences. Cite any theorem you use by name, if possible, and make sure to check its hypotheses. No electronic devices are permitted.

1. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given below. Determine whether $f$ is continuous at $(0,0)$. Justify your answer.

$$f(x, y) := \begin{cases} \frac{x^3 y}{(x^2 + y^2)^2} & \text{if } (x, y) \neq (0,0), \\ 0 & \text{if } (x, y) = (0,0). \end{cases}$$

Let $z_n = \left(\frac{1}{n}, \frac{1}{n}\right)$ for $n \in \mathbb{N}$. Then $z_n \to (0,0)$ but

$$f(z_n) = \frac{\frac{1}{n^3} \cdot \frac{1}{n}}{\left(\frac{1}{n^2} + \frac{1}{n^2}\right)^2} = \frac{\frac{1}{n^4}}{\frac{4}{n^4}} = \frac{1}{4} \not\to 0 = f(0,0).$$

Hence by the sequential characterization of continuity $f$ is not continuous at $(0,0)$. 

2. Let \( f : \mathbb{R}^2 \to \mathbb{R} \) be defined by \( f(x, y) = xy \). Use the definition to prove that \( f \) is differentiable at \((1, 2)\) and that its differential is given by \( df(1, 2)(h_1, h_2) = 2h_1 + h_2 \) for all \((h_1, h_2) \in \mathbb{R}^2\).

Let \( a = (1, 2), \ h = (h_1, h_2) \) and \( L(h) = 2h_1 + h_2 \). Then

\[
\frac{f(a + h) - f(a) - L(h)}{h} = \frac{(1+h_1)(2+h_2) - 2 - 2h_1 - h_2}{h} = \frac{2 + 2h_1 + h_2 + h_1h_2 - 2 - 2h_1 - h_2}{h} = h_1h_2
\]

So we have

\[
\lim_{h \to 0} \frac{f(a + h) - f(a) - L(h)}{\|h\|} = \lim_{(h_1, h_2) \to (0, 0)} \frac{h_1h_2}{\sqrt{h_1^2 + h_2^2}}.
\]

Now, since \( |h_1| \leq \sqrt{h_1^2 + h_2^2} \) for \( i = 1, 2 \), we have

\[
\frac{|h_1h_2|}{\sqrt{h_1^2 + h_2^2}} \leq \frac{h_1^2 + h_2^2}{\sqrt{h_1^2 + h_2^2}} = \sqrt{h_1^2 + h_2^2}.
\]

Since \( \lim_{(h_1, h_2) \to (0, 0)} \sqrt{h_1^2 + h_2^2} = 0 \) it follows by the Squeeze Th that

\[
\lim_{h \to 0} \frac{f(a + h) - f(a) - L(h)}{\|h\|} = 0
\]

Hence \( f \) is differentiable at \((1, 2)\) with \( df(1, 2)(h_1, h_2) = 2h_1 + h_2 \).
3. Let $G : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $G(x, y) = (y \ln(1 + x^2), xe^{-y})$. Show that $G$ is a $C^1$ transformation and find $[dG(a, b)]$, its Jacobian matrix at $(a, b) \in \mathbb{R}^2$.

We have

$$\frac{\partial g_1}{\partial x} (x, y) = \frac{2xy}{1+x^2} \quad \frac{\partial g_1}{\partial y} (x, y) = \ln(1+x^2)$$

$$\frac{\partial g_2}{\partial x} (x, y) = e^{-y} \quad \frac{\partial g_2}{\partial y} (x, y) = -xe^{-y}$$

Since the first partials of $g_1, g_2$ are continuous on $\mathbb{R}^2$, we may conclude that $G$ is $C^1$ on $\mathbb{R}^2$. Moreover,

$$[dG(a, b)] = \begin{pmatrix}
\frac{2ab}{1+a^2} & \ln(1+a^2) \\
\frac{ab}{1+a^2} e^{-b} & -ae^{-b}
\end{pmatrix}$$
4. Let $F : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be a smooth transformation and let $H : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be defined by $H(x, y) := F(x^2 - y^2, 2xy)$ for all $(x, y) \in \mathbb{R}^2$. Prove that $H$ is differentiable at $(2, -1)$ and find $[dH(2, -1)]$ given that $[dF(3, -4)] = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

Define $G : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ by $G(x, y) = (x^2 - y^2, 2xy)$. Then since

$$\frac{\partial g_1}{\partial x} = 2x, \quad \frac{\partial g_1}{\partial y} = -2y, \quad \frac{\partial g_2}{\partial x} = 2y, \quad \frac{\partial g_2}{\partial y} = 2x$$

are all cts, $G$ is diff on $\mathbb{R}^2$ and in particular at $(2, -1)$.

Since $H(x, y) = F(G(x, y))$ for $(x, y) \in \mathbb{R}^2$, $H = F \circ G$. Moreover, $F$ is diff at $(3, -4) = G(2, -1)$ and so by the Chain Rule $H$ is diff at $(2, -1)$ and

$$[dH(2, -1)] = [dF(3, -4)] [dG(2, -1)]$$

$$= \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 4 - 2 \\ -2 & 4 \end{pmatrix} = \begin{pmatrix} 4a - 2b & 2a + 4b \\ 4c - 2d & 2c + 4d \end{pmatrix}$$
5. Let \( f : \mathbb{R}^2 \to \mathbb{R} \) be defined by \( f(x, y) := x^2 y - 2x^2 - y^2 + 2y \). Show that \( a = (0, 1) \) is a critical point of \( f \), that is, either \( f \) is differentiable at \( a \) and \( df(a) = 0 \) or \( f \) is not differentiable at \( a \). Determine if a local extremum occurs at \( a \) and, if so, whether it is a local maximum or a local minimum.

We have:
\[
\frac{\partial f}{\partial x} = 2xy - 4x = 2x(y-2),
\]
\[
\frac{\partial f}{\partial y} = x^2 - 2y + 2.
\]

Note that these partials are continuous on \( \mathbb{R}^2 \) and so \( f \) is differentiable everywhere, in particular at \( a = (0, 1) \).

We have:
\[
\frac{\partial f}{\partial x}(a) = 2 \cdot 0 \cdot (1-2) = 0 \quad \text{and}\]
\[
\frac{\partial f}{\partial y}(a) = 0^2 - 2 \cdot 1 + 2 = 0.
\]

Thus \( df(a) = 0 \) and so \( a \) is a c.p.

We have:
\[
\frac{\partial^2 f}{\partial x^2}(x, y) = 2y - 4, \quad \frac{\partial^2 f}{\partial x \partial y}(x, y) = \frac{\partial^2 f}{\partial y \partial x}(x, y) = 2x,
\]
\[
\frac{\partial^2 f}{\partial y^2}(x, y) = -2.
\]

So \( \Delta = \det d^2 f(a) = \begin{vmatrix} -2 & 0 \\ 0 & -2 \end{vmatrix} = 4 > 0 \) so \( a \) is a local extremum occurs at \( a \). Furthermore, we have:
\[
\frac{\partial^2 f}{\partial x^2}(0, 1) = -2 < 0 \quad \text{thus, } f(0, 1) = 1 \text{ is a local max.}
\]
XC (Extra Credit 4 pts) Let $A = (a_{ij})$ be a nonzero $q \times p$ matrix. Prove that

$$
\|A\|_M = \sup \{Ax \cdot y : \|x\| = \|y\| = 1, x \in \mathbb{R}^p, y \in \mathbb{R}^q \}.
$$

Set $K = \sup \|Ax\|_M$.

Let $x \in \mathbb{R}^p, y \in \mathbb{R}^q$ s.t. $\|x\| = \|y\| = 1$. The since $\|Ax\| \leq \|A\| M$, we have, by the Cauchy-Schwarz inequality

$$(\#) \quad Ax \cdot y \leq \|Ax\| \cdot \|y\| \leq \|A\| M.$$

Let $S = \{x \in \mathbb{R}^p : \|x\| = 1\}$. Then since $S$ is compact and the map $x \mapsto \|Ax\|$ is cts on $S$, $\exists x_0 \in S$ s.t.

$$
\|A\|_M = \sup_{x \in S} \|Ax\| = \|Ax_0\|.
$$

Then since $Ax_0 \neq 0$, we may define $y_0 \in \mathbb{R}^q$ by

$$
\frac{1}{\|Ax_0\|} A x_0.
$$

Then $\|y_0\| = 1$ and $A x_0 \cdot y_0 = \frac{1}{\|Ax_0\|} A x_0 \cdot A x_0 = \|Ax_0\| = \|A\| M$.

Hence, $\|A\|_M \leq K$ and by $(\#)$ above $K = \|A\|_M$. Combining the two we have $K = \|A\|_M$ as desired.