The Final Exam will be comprehensive with some emphasis on the material not covered on the two tests (i.e. sections 6.4, 6.5, 7.1). You may bring a formula sheet (two sides of one sheet of paper). There will be a review Monday and, if there is interest, another review on Friday 2:00-3:30 (Dec 15) in dm3315. Please review the two tests, quizzes, the earlier review sheets and homework. Here are some sample questions for material from 6.4, 6.5, 7.1.

(1) Find an orthonormal basis for \( W = \text{Span}\{x_1, x_2\} \) (first use the Gram-Schmidt process and then normalize the resulting vectors).

\[
x_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \quad x_2 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}.
\]

(2) Find a least squares solution of \( Ax = b \) by first finding the normal equations and then solving them. Next, find the least squares error \( \|b - A\hat{x}\| \).

\[
A = \begin{bmatrix} 1 & 3 \\ 1 & -1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 5 \\ 2 \\ 0 \\ 3 \end{bmatrix}, \quad \hat{x} = (1, 1), \quad \|b - A\hat{x}\| = \sqrt{6}
\]

(3) Consider the least-squares problem \( Ax = b_i \), \( i = 1, 2 \), where \( A, b_1 \) and \( b_2 \) are as given:

\[
A = \begin{bmatrix} 1 & -1 & -1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \\ 1 & 1 & 1 \end{bmatrix}, \quad b_1 = \begin{bmatrix} 2 \\ 0 \\ 3 \\ 4 \end{bmatrix}, \quad b_2 = \begin{bmatrix} 0 \\ 2 \\ 3 \\ 4 \end{bmatrix}
\]

(a) Use the fact that the columns of \( A \) are orthogonal to find \( \tilde{b}_i \), \( i = 1, 2 \), the orthogonal projection of \( b \) onto \( \text{Col} \ A \).

\[
\tilde{b}_1 = b_1 = (1, 2, 3, 4), \quad \tilde{b}_2 = \frac{1}{4}(1, 7, 11, 17)
\]

(b) Find a least-squares solution of \( Ax = b \).

\[
\tilde{x}_1 = \frac{1}{2}(5, 2, 1), \quad \tilde{x}_2 = \frac{1}{2}(9, 5, 3)
\]

(4) Find the QR factorization of \( A \) and the projection matrix onto \( \text{Col} \ A \).

\[
A = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 0 & -1 \\ 1 & 0 & 0 \end{bmatrix}
\]

\[
Q = \frac{1}{2} \begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & 1 \\ 1 & -1 & -1 \end{bmatrix}
\]

(5) Orthogonally diagonalize each of the following, i.e., express it as \( PDPT^T \), where \( D \) is a diagonal matrix and \( P \) is an orthogonal matrix \( P \).

\[
A = \frac{4}{5} \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} - \frac{1}{5} \begin{bmatrix} 1 & -2 \\ -2 & 1 \end{bmatrix}; \quad B = \frac{2}{3} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} - \frac{1}{3} \begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix}
\]

Find the spectral decompositions

\[
A = \begin{bmatrix} 0 & 2 \\ 2 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}.
\]

(6) Use the Gram-Schmidt process to find an orthonormal basis for the the column space of the matrix \( A \) where

\[
A = \begin{bmatrix} 3 & -5 & 1 \\ 1 & 1 & 1 \\ 1 & 5 & -2 \\ 3 & -7 & 8 \end{bmatrix}
\]

Find the QR factorization for \( A \).

\[
Q = \frac{1}{\sqrt{20}} \begin{bmatrix} 3 & 1 & -3 \\ 1 & 3 & 1 \\ -1 & 3 & 1 \\ 3 & -1 & 3 \end{bmatrix}
\]