Coverage for Test II will include sections §4.2–4.6, 5.1–5.3, 5.5, 5.7, 6.1–6.3. There will be a review Monday and if there is interest a supplementary review Tuesday afternoon. You may bring a formula sheet to the test (one side only; no worked problems). Go over past HW and quizzes. Here is a list of some sample questions.

1. Find a matrix \( A \) so that the \( W = \text{Col} \ A \). Find a basis for \( W \).

\[
W = \left\{ \begin{bmatrix} 2s + t \\ s - 2t \\ 3s - t \end{bmatrix} : s \text{ and } t \text{ are real} \right\}.
\]

2. With \( A \) as follows find bases for \( \text{Col} \ A, \text{Nul} \ A \) and \( \text{Row} \ A \). Find the dimension of each space and determine rank \( A \).

\[
A = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 2 & 3 & 1 & 3 \\ 1 & 2 & 2 & 4 \end{bmatrix}
\]

\( \{a_1, a_2, a_3\}, \{(−2, 1, 0, 1), (0, 1, 1, 1), (0, 0, 1, 2)\}; 3, 1, 3, 3 \)

3. Find a basis for \( \mathbb{P}_3 \) and \( \dim \mathbb{P}_3 \) where

\[ \mathbb{P}_3 = \{a_0 + a_1 t + a_2 t^2 + a_3 t^3 : a_0, a_1, a_2, a_3 \text{ real} \}. \]

4. Let \( A \) be a \( 5 \times 7 \) matrix with three pivot columns. Find \( \dim \text{Nul} \ A, \text{rank} \ A, \dim \text{Nul} \ A^T \) and \( \text{rank} \ A^T \).

5. The set \( B = \{1 + t + t^2, t - 2t^2, t^3\} \) forms a basis for \( \mathbb{P}_2 \). Find the coordinate vector of \( p(t) = 4 + 3t - t^2 \) relative to \( B \).

\[ |p(t)|_B = [1 \ -1 \ -7]^T \]

6. For each of the following matrices, find the characteristic equation, the eigenvalues and a basis for each eigenspace:

\[
A: \lambda - 3)(\lambda - 5) = 0; \ E_1 \ basis \ \{1 [-2] \ 1]^T \}; \ E_2 \ basis \ \{1 [-1]^T \};
\]

\[
C: -\lambda(\lambda - 2)^2; \ E_0 \ basis \ \{1 [0 \ -1]^T \}; \ E_2 \ basis \ \{0 [1 \ 0]^T \}; \ [1 \ 0 \ 1]^T \}
\]

7. For each of the above matrices find if possible a diagonal matrix \( D \) and an invertible matrix \( P \) such that \( A = PDPP^{-1} \). Check your answer by showing that \( AP = PD \).

8. With \( A \) as below, find a diagonal matrix \( D \) and an invertible matrix \( P \) such that \( A = PDPP^{-1} \). Use this factorization to compute \( A^8 \).

\[
A = \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix}
\]

\[
D = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}, \ P = \begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix}
\]

9. Find the eigenvalues and corresponding eigenvectors in \( \mathbb{C}^2 \) for \( A \).

\[
\lambda = 3 \pm 2i, \ v = [-4 \ 2 \mp 3]^T
\]

10. With \( u_1, u_2 \) as given below, show that \( \{u_1, u_2\} \) forms an orthogonal set. Let \( W = \text{Span}\{u_1, u_2\} \) and find a matrix \( A \) such that \( \text{Nul} \ A = W^⊥ \). Find a nonzero vector \( u_3 \) in \( W^⊥ \). Does \( \{u_1, u_2, u_3\} \) form an orthogonal basis for \( \mathbb{R}^3 \)?

\[
u_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \ u_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix} \]

\[
u_3 = \begin{bmatrix} 1 \ 0 \ -1 \end{bmatrix}, \ \text{yes}
\]

Find the projection \( \tilde{y} = \text{proj}_W y \) and show that \( y - \tilde{y} = \text{proj}_{W^⊥} y \) where

\[
y = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}
\]

\[
\tilde{y} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}
\]

Find the distance from \( y \) to \( W \) and the distance from \( y \) to \( W^⊥ \).

(11) Consider the vectors:

\[
u_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \ u_2 = \begin{bmatrix} 1 \\ 4 \\ -1 \end{bmatrix}, \ u_3 = \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix}, \ x = \begin{bmatrix} 3 \\ 5 \\ 7 \end{bmatrix}.
\]

a. Show that \( \{u_1, u_2, u_3\} \) forms an orthogonal basis for \( \mathbb{R}^3 \).

b. By normalizing these vectors find an orthonormal basis for \( \mathbb{R}^3 \) and show that the \( 3 \times 3 \) matrix \( U \) obtained from this basis satisfies \( U^TU = I_3 \).

c. Use orthogonality to express \( x \) as a linear combination of \( u_1, u_2, u_3 \).

\[
x = 5u_1 + \frac{8}{5}u_2 + \frac{13}{7}u_3.
\]
With $A$ given below find the general solution to the vector differential equation $\mathbf{x}' = A\mathbf{x}$ and then find the solution which satisfies the initial condition $\mathbf{x}(0) = (5, -1)$. 

$$x(t) = c_1 e^{2t} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}; \ c_1 = 1, c_2 = -2$$

$A = \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix}$

With $A$ given below find two independent complex solutions to the vector differential equation $\mathbf{x}' = A\mathbf{x}$. Then find the general real solution.

$$e^{(2+i)t} \begin{bmatrix} -1 + i \\ 1 \\ -1 - i \\ 1 \end{bmatrix}, e^{(2-i)t} \begin{bmatrix} -1 - i \\ 1 \\ -1 + i \\ 1 \end{bmatrix}; \ y(t) = c_1 e^{2t} \begin{bmatrix} \cos t - \sin t \\ \cos t \\ \cos t - \sin t \\ \cos t \end{bmatrix} + c_2 e^{2t} \begin{bmatrix} \cos t + \sin t \\ \cos t \\ \cos t + \sin t \\ \cos t \end{bmatrix}$$

$A = \begin{bmatrix} 1 & -2 \\ 1 & 3 \end{bmatrix}$

Explain why the given set $W$ is a subspace of $\mathbb{R}^3$. Find a matrix $A$ so that $W = \text{Col } A$.

$$W = \left\{ \begin{bmatrix} r + 5s - t \\ 2r + 4s \\ 3s - t \end{bmatrix} : r, s, t \text{ real} \right\}$$

Find a basis for $\text{Nul } A$, $\text{Col } A$ and $\text{Row } A$, where the matrix $A$ and its row reduced echelon form $B$ are given below. How did you find the basis for $\text{Col } A$? Determine $\dim \text{Col } A$, $\dim \text{Nul } A$ and rank $A$.

$$A = \begin{bmatrix} 1 & 1 & -5 & 1 & 4 \\ 2 & 1 & -7 & 3 & 8 \\ 2 & -3 & 5 & 0 & 1 \\ 0 & 1 & -3 & 1 & 2 \end{bmatrix}, \quad B = \text{rref } A = \begin{bmatrix} 1 & 0 & -2 & 0 & 2 \\ 0 & 1 & -3 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$\{(2,3,1,0,0),(-2,-1,0,-1,1),\{(1,2,2,0),(1,1,-3,1),(1,3,0,1)\}, \{(1,0,-2,0,2),(0,1,-3,0,1),(0,0,0,1,1)\}; 3,2,3$

Find $[\mathbf{x}]_B$, the coordinate vector of $\mathbf{x}$ relative to the basis $B = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$:

$$\mathbf{x} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \quad \mathbf{b}_1 = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}, \quad \mathbf{b}_2 = \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}, \quad \mathbf{b}_3 = \begin{bmatrix} -2 \\ 6 \\ -5 \end{bmatrix}$$

Is $\lambda = 3$ an eigenvalue of the following matrix? If so, find a corresponding eigenvector (please indicate any elementary row operations you use).

$$A = \begin{bmatrix} 4 & 0 & 2 \\ 2 & 4 & 3 \\ 3 & 2 & 7 \end{bmatrix}$$

Yes, $[\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}]^T$

Find the characteristic polynomial and the eigenvalues for the following matrix:

$$p(\lambda) = -(\lambda + 1)(\lambda - 2)(\lambda - 4)$$

With $A$ as below, find a diagonal matrix $D$ and an invertible matrix $P$ such that $A = PD P^{-1}$.

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 3 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

Find the eigenvalues and corresponding eigenvectors in $\mathbb{C}^2$ for the matrix

$$A = \begin{bmatrix} 3 & -2 \\ 1 & 1 \end{bmatrix}$$

With $\mathbf{y}$ and $\mathbf{u}$ as given below. Write $\mathbf{y}$ as the sum of two vectors, one in $\text{Span}\{\mathbf{u}\}$ and one orthogonal to $\mathbf{u}$.

$$\mathbf{y} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \quad \mathbf{u} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

Show that $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ form an orthogonal basis for $\mathbb{R}^3$. Use orthogonality to express $\mathbf{x}$ as a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$.

$$\mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \quad \mathbf{u}_3 = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}.$$  

With $\mathbf{y}, \mathbf{u}_1, \mathbf{u}_2$ as given below, find a vector $\hat{\mathbf{y}}$ in $W = \text{Span}\{\mathbf{u}_1, \mathbf{u}_2\}$ and a vector $\mathbf{z}$ in $W^\perp$ so that $\mathbf{y} = \hat{\mathbf{y}} + \mathbf{z}$. Find the distance from $\mathbf{y}$ to $W$ and the point in $W$ which lies closest to $\mathbf{y}$.

$$\mathbf{y} = \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}, \quad \mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}.$$