Graded $K$-theory and Graph Algebras

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A $C^*$-algebra $A$ endowed with an automorphism of order two is said to be $\mathbb{Z}_2$-graded or simply graded. Forty years ago Kasparov introduced $KK$, a bivariate $K$-functor for graded $C^*$-algebras, that revolutionized the study of $K$-theory for trivially graded $C^*$-algebras. The work [KPS] reported on here grew out of an attempt by David Pask, Aidan Sims and me to understand the $K$-theory of graded graph $C^*$-algebras. We proved graded versions of Pimsner’s six-term exact sequences for $KK$-groups associated to Cuntz-Pimsner algebras and derived the existence of an exact sequence for computing the $K$-theory of graded graph $C^*$-algebras that generalizes the usual one for computing the $K$-theory of trivially graded graph $C^*$-algebras. We also proved a graded version of the Pimsner-Voiculescu six-term exact sequence.
Graded $C^*$-algebras

A $C^*$-algebra $A$ equipped with an automorphism $\alpha$ of order two is said to be *graded* and $\alpha$ is called the grading automorphism or simply the grading. Denote the induced grading on $\mathcal{M}(A)$ by $\alpha$. An element $a \in A$ is said to be *homogeneous* of degree $\partial a = i$ where $i \in \mathbb{Z}_2$ if $\alpha(a) = (-1)^{\partial a} a$. Every $a \in A$ may be expressed uniquely as a sum $a = a_0 + a_1$ where $\partial a_i = i$. Thus $A = A_0 \oplus A_1$ where $A_i$ consists of all $a \in A$ of degree $i$.

We say that $A$ is *inner graded* if there is a self-adjoint unitary $u \in \mathcal{M}(A)$ s.t. $\alpha(a) = uau$ for all $a \in A$.

The first (complex) Clifford algebra is the $C^*$-algebra $C_1 = \mathbb{C} \oplus \mathbb{C}$ endowed with the grading $\alpha(x, y) = (y, x)$.

There is a graded tensor product $A \widehat{\otimes} B$ of graded $C^*$-algebras s.t. if $a, a' \in A$ and $b, b' \in B$ are homogeneous elements

$$\partial(a \widehat{\otimes} b) = \partial a + \partial b, \quad (a \widehat{\otimes} b)(a' \widehat{\otimes} b') = (-1)^{\partial b \partial a'} aa' \widehat{\otimes} bb'.$$

Note that $C_2 := C_1 \widehat{\otimes} C_1 \cong M_2(\mathbb{C})$ with inner grading determined by $u = \text{diag}(1, -1)$. More generally, $C_{n+1} := C_n \widehat{\otimes} C_1$. 

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Graph C*-algebras owe their origin to Cuntz algebras and more generally Cuntz-Krieger algebras.

Let $E = (E^1, E^0, r, s)$ be a directed graph with vertices $E^0$, edges $E^1$ and maps $r, s : E^1 \to E^0$. Suppose $0 < |r^{-1}(v)| < \infty$ for all $v$. Let $C^*(E)$ denote the universal C*-algebra generated by an orthogonal set of projections $\{p_v : v \in V\}$ and a family of partial isometries $\{t_e : e \in E\}$ satisfying

i) $t_e^* t_e = p_{s(e)}$ for every $e \in E^1$,

ii) $p_v = \sum_{r(e) = v} t_e t_e^*$ for every $v \in E^0$.

Given a map $\delta : E^1 \to \mathbb{Z}_2$, the universality of $C^*(E)$ ensures the existence of a grading $\alpha = \alpha^\delta$ such that for all $v \in E^0, e \in E^1$

$$\alpha(p_v) = p_v \quad \text{and} \quad \alpha(t_e) = (-1)^{\delta(e)} t_e.$$ 

If $\delta(e) = 1$ for all $e \in E^1$, $\alpha$ is called the standard grading.
Some examples

Consider the graph $B_n$ with $B_n^0 := \{v\}$ and $B_n^1 := \{e_1, \ldots, e_n\}$.

Since $B_1$ has a one edge and one vertex the definition implies that $C^*(B_1) \cong C(\mathbb{T})$. If $n \geq 2$, $C^*(B_n) \cong \mathcal{O}_n$.

Define the directed cycle $C_n$ with $C_n^0 = \{v_1, \ldots, v_n\}$ and $C_n^1 = \{e_1, \ldots, e_n\}$ such that $r(e_i) = v_i$ and $s(e_i) = v_{i+1}$ if $i < n$ and $s(e_n) = v_1$. Then $C^*(B_n) \cong M_n(C(\mathbb{T}))$. The center is generated by $\sum_i t_{e_i} \cdots t_{e_n} t_{e_1} \cdots t_{e_{i-1}}$.

Define the graph $\Omega$ by $\Omega^0 = \Omega^1 := \mathbb{N}$ with $s(n) = n + 1$ and $r(n) = n$. Then $C^*(\Omega) \cong \mathcal{K}(\ell^2(\mathbb{N}))$. 
Graded $K$-theory

We use Kasparov’s $KK$ bifunctor (see [K], [B, 17.3]) to define the $K$-theory of a graded $C^*$-algebra. See [vD, H] for other approaches.

**Definition**

Let $(A, \alpha)$ be a $\sigma$-unital graded $C^*$-algebra. Define the graded $K$ groups of $A$ as follows:

$$K_0^{gr}(A, \alpha) := KK(\mathbb{C}, A),$$

$$K_1^{gr}(A, \alpha) := KK(\mathbb{C}, A \widehat{\otimes} C_1).$$

When $\alpha$ is understood from context write $K_i^{gr}(A)$ for $K_i^{gr}(A, \alpha)$.

Note: We have $K_*^{gr}(\mathbb{C}) = (\mathbb{Z}, 0)$ and $K_*^{gr}(C_1) = (0, \mathbb{Z})$.

If $A$ is inner graded, then $K_i^{gr}(A) = K_i(A)$.

Skandalis has shown that under mild hypotheses the six-term exact sequence for extensions also holds for graded $K$-theory (see [S]).
Let $E$ be a graph s.t. $0 < |r^{-1}(v)| < \infty$ for all $v$. Let $\delta : E^1 \to \mathbb{Z}_2$ be a map and let $\alpha = \alpha^\delta$. Let $A^\delta_E$ be the $E^0 \times E^0$ matrix defined by

$$A^\delta_E(v, w) = \sum_{e \in vE^1 w} (-1)^{\delta(e)}.$$

**Theorem**

The following sequence is exact.

$$0 \to K_1^{gr}(C^*(E), \alpha) \to \mathbb{Z}E^0 \xrightarrow{1-(A^\delta_E)^t} \mathbb{Z}E^0 \to K_0^{gr}(C^*(E), \alpha) \to 0$$

**Example (a)**

Let $C_n$ be the directed cycle as above and let $\delta(e_i) = 1$ for $i = 1, \ldots, n$. By the above theorem, $K_1^{gr}(C^*(C_n), \alpha) \cong (\mathbb{Z}, \mathbb{Z})$ if $n$ is even and $K_1^{gr}(C^*(C_n), \alpha) \cong (\mathbb{Z}_2, 0)$ if $n$ is odd.
Example (b)

Let $B_n$ be the graph with one vertex and $n$ edges and let $\delta : B_n^1 \to \{0, 1\}$ be a map. Set $p := |\delta^{-1}(1)|$ and $q := |\delta^{-1}(0)|$. Note that $C^*(B_n) \cong \mathbb{O}_n$ and $A^\delta_{B_n}$ is a $1 \times 1$ matrix with entry $q - p$. Therefore we have (cf. [H, 4.11])

$$K^\text{gr}_*(\mathbb{O}_n) \cong \begin{cases} (\mathbb{Z}|1+p-q|, 0) & \text{if } 1 + p - q \neq 0, \\ (\mathbb{Z}, \mathbb{Z}) & \text{otherwise}. \end{cases}$$

Example (c)

We define a graph $E$ as follows: set $E^0 := \{v_n : n \in \mathbb{Z}\}$ and $E^1 := \{e_n, f_n : n \in \mathbb{Z}\}$; the range and source maps are given by $r(e_n) = r(f_n) = v_n$ and $s(e_n) = v_{n+1}$ and $s(f_n) = v_n$ for $n \in \mathbb{Z}$. Define $\delta : E^1 \to \mathbb{Z}_2$ by $\delta(e_n) = \delta(f_n) = 1$ for all $n$. Then $A^\delta_E(v_m, v_n) = -1$ if $n = m, m + 1$ and 0 otherwise; by a routine computation, $K^\text{gr}_0(C^*(E), \alpha) = \mathbb{Z}[\frac{1}{2}]$ and $K^\text{gr}_1(C^*(E), \alpha) = 0$. 
Let \((A, \alpha)\) be a \(\sigma\)-unital graded \(C^*\)-algebra and let \(\gamma \in \text{Aut}(A)\) s.t. 
\(\gamma \alpha = \alpha \gamma\). There are two natural gradings \(\beta^k\) for \(k = 0, 1\) on the 
crossed product \(A \rtimes_\gamma \mathbb{Z}\). Let \(j : A \to A \rtimes_\gamma \mathbb{Z}\) denote the natural 
inclusion and let \(u \in \mathcal{M}(A)\) be the canonical unitary such that 
\(j(\gamma(a)) = uj(a)u^*\) for all \(a \in A\). Then for \(k = 0, 1\) the gradings are 
determined by the formulae: 
\(\beta^k(u) = (-1)^k u, \beta^k(j(a)) = j(\alpha(a))\) for all \(a \in A\).

**Theorem**

*The following six-term exact sequence is exact for \(k = 0, 1\).*

\[
\begin{align*}
K_0^{gr}(A \rtimes_\gamma \mathbb{Z}, \beta^k) & \xrightarrow{id - (-\alpha^*)_k \gamma_*} K_0^{gr}(A, \alpha) \xrightarrow{j_*} K_0^{gr}(A \rtimes_\gamma \mathbb{Z}, \beta^k) \\
K_1^{gr}(A \rtimes_\gamma \mathbb{Z}, \beta^k) & \xleftarrow{j_*} K_1^{gr}(A, \alpha) \xleftarrow{id - (-\alpha^*)_k \gamma_*} K_1^{gr}(A, \alpha)
\end{align*}
\]
Example (d)

Take $A = \mathbb{C}$ with the trivial grading and let $\gamma$ be the identity automorphism. Then $A \rtimes_{\gamma} \mathbb{Z} \cong C^*(\mathbb{Z}) \cong C(\mathbb{T})$ and $\beta^0$ is trivial but not $\beta^1$. Under the above isomomorphism $\beta^1(f)(z) = f(-z)$. In this case we have $\text{id} - (-\alpha_*)^k \gamma_* = 2 \text{id}$. Since $K^\text{gr}_0(A, \alpha) = K_0(\mathbb{C}) \cong \mathbb{Z}$ and $K^\text{gr}_1(A, \alpha) = K_1(\mathbb{C}) \cong 0$, we have

$$K^\text{gr}_0(C(\mathbb{T}), \beta^1) \cong \mathbb{Z}_2 \quad \text{and} \quad K^\text{gr}_0(C(\mathbb{T}), \beta^1) = 0.$$ 

Example (e)

Now let $E$ and $\delta$ be as in Example (c) above and let $\gamma$ be the automorphism of $C^*(E)$ such that $\gamma(p_{v_n}) = p_{v_{n+1}}$, $\gamma(s_{e_n}) = s_{e_{n+1}}$, $\gamma(s_{f_n}) = s_{f_{n+1}}$ for all $n$. Note that the map induced by the grading on $K^\text{gr}_i(C^*(E), \alpha)$ is the identity (see [KPS, 8.8]) and the map induced by $\gamma_*$ is multiplication by $-2$. So if we take the grading $\beta^1$ on $A \rtimes_{\gamma} \mathbb{Z}$ the map $\text{id} - (-\alpha_*) \gamma_*$ on $K^\text{gr}_0(C^*(E), \alpha)$ is a bijection. Hence $K^\text{gr}_i(A \rtimes_{\gamma} \mathbb{Z}, \beta^1) = 0$ for $i = 0, 1$. 

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[B] B. Blackadar,  

[vD] A. van Daele,  

[H] U. Haag,  

[K] G.G. Kasparov,  

[KPS] Kumjian, Pask and Sims,  

[P] M. Pimsner,  

[S] G. Skandalis,  
*Exact sequences for the Kasparov groups of graded algebras*, 1985.
Questions?
Let \((A, \alpha)\) be a \(\sigma\)-unital graded \(C^*\)-algebra and let \(X\) be a countably generated graded correspondence over \(A\). There is an element \([X] \in KK(A, A)\) and a grading \(\alpha^X\) on the Cuntz-Pimsner algebra \(\mathcal{O}_X\) compatible with the map \(i: A \to \mathcal{O}_X\). Since \(K^\text{gr}_0(A) = KK(\mathbb{C}, A)\), the product with \([id_A] - [X]\) yields an endomorphism on \(K^\text{gr}_0(A)\) denoted \(\hat{\otimes}_A([id_A] - [X])\) (cf. [P, 4.9]).

**Theorem (Pimsner)**

*With notation as above suppose that left action of \(A\) on \(X\) is injective and by compacts. Then the following is exact.*

\[
\begin{align*}
K^\text{gr}_0(A, \alpha) & \xrightarrow{\hat{\otimes}_A([id_A] - [X])} K^\text{gr}_0(A, \alpha) & i_* \to K^\text{gr}_0(\mathcal{O}_X, \alpha^X) \\
K^\text{gr}_1(\mathcal{O}_X, \alpha^X) & \xleftarrow{i_*} K^\text{gr}_1(A, \alpha) & \hat{\otimes}_A([id_A] - [X]) \xleftarrow{} K^\text{gr}_1(A, \alpha)
\end{align*}
\]
If there is time, here’s one more example.

**Example (f)**

We define a graph $E$ as follows: set $E^0 := \{v_n : n \in \mathbb{Z}\}$ and $E^1 := \{e_n, f_n : n \in \mathbb{Z}\}$; the range and source maps are given by $r(e_n) = r(f_n) = v_n$ and $s(e_n) = s(f_n) = v_{n+1}$ for $n \in \mathbb{Z}$. Define $\delta : E^1 \to \mathbb{Z}_2$ by $\delta(e_n) = 0$ and $\delta(f_n) = 1$ for all $n$. Then $A^\delta_E$ is zero and, hence, the map $\mathbb{Z}E^0 \to \mathbb{Z}E^0$ is the identity in the exact sequence above. Thus, $K_i^{gr}(C^*(E), \alpha) = 0$ for $i = 0, 1$. Note that $C^*(E)$ is Morita equivalent to the UHF algebra $M_{2\infty}$.