Homology and cohomology of higher rank graphs

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Research on higher rank graphs (or \( k \)-graphs) and their \( C^* \)-algebras was inspired by the work of Robertson and Steger on \( C^* \)-algebras arising from certain group actions on buildings.

If a \( k \)-graph \( \Lambda \) satisfies mild hypotheses, there is a path groupoid \( \mathcal{G}_\Lambda \) such that \( C^*(\Lambda) \cong C^*(\mathcal{G}_\Lambda) \).

Recent work in this area has focused on the homology and cohomology of higher rank graphs and the twisted \( k \)-graph \( C^* \)-algebras arising from 2-cocycles with values in \( \mathbb{T} \).

Two distinct cohomology theories, cubical and categorical, have emerged. Elizabeth Gillaspy and Jianchao Wu have shown that the two are isomorphic.

Much of the material in this talk is based on joint work with David Pask, Aidan Sims, both at the University of Wollongong, and Elizabeth Gillaspy, at the University of Montana.
Higher rank graphs

A *small category* is one for which the morphisms constitute a set. Given a small category $\mathcal{C}$ we identify $\text{Obj} \mathcal{C}$, the set of objects, with $\mathcal{C}^0$, the set of identity morphisms.

A pair of morphisms $(\lambda, \mu)$ is *composable* if $s(\lambda) = r(\mu)$. The composition is denoted $\lambda \mu$.

Let $k \in \mathbb{N} := \{0, 1, 2, \ldots \}$. Regard $\mathbb{N}^k$ as a monoid with identity 0 and generators $\varepsilon_1, \ldots, \varepsilon_k$. Set $1_k := \varepsilon_1 + \cdots + \varepsilon_k$.

**Definition**

Let $\Lambda$ be a small category and $d : \Lambda \rightarrow \mathbb{N}^k$ be a functor. $(\Lambda, d)$ is a *higher rank graph* or a *$k$-graph* if it satisfies the following *factorization property*:

For every $\lambda \in \Lambda$ and $m, n \in \mathbb{N}^k$ such that $d(\lambda) = m + n$, there exist unique $\mu, \nu \in \Lambda$ such that

$$\lambda = \mu \nu, \quad d(\mu) = m \quad \text{and} \quad d(\nu) = n.$$
For $n \in \mathbb{N}^k$ set $\Lambda^n := d^{-1}(n)$; observe that $\lambda \in \Lambda^0$ iff it is an identity. Each $\nu \in \Lambda^0$ is called a vertex and an element $\lambda \in \Lambda^{\varepsilon_i}$ is called an edge.

We often assume that $\Lambda$ is row-finite and source-free, that is, for all $\nu \in \Lambda^0, n \in \mathbb{N}^k$, $\nu \Lambda^n := r^{-1}(\nu) \cap \Lambda^n$ is finite and nonempty.

Let $\Lambda$ be a $k$-graph.

- If $k = 1$, then $\Lambda$ is the path category of a directed graph.
- If $k \geq 2$, think of $\Lambda$ as generated by $k$ directed graphs of different colors that share the same set of vertices $\Lambda^0$.

A morphism of $k$-graphs is a functor which preserves the degree map.

A quasimorphism is a functor which is compatible with a given morphism between the degree monoids.

Note that $T_k := \mathbb{N}^k$ is a $k$-graph with one vertex. It may be regarded as the $k$-graph analog of a torus.

Let $C_n$ denote the directed cycle with $n$ vertices viewed as a 1-graph.
Some examples

Example of a 2-graph $\Lambda$: Only $\Lambda^0$, $\Lambda^{\varepsilon_1}$ and $\Lambda^{\varepsilon_2}$, are shown.

Note that $\Lambda \cong C_2 \times C_1$.

Let $\Omega_k := \{(m, n) \in \mathbb{N}^k \times \mathbb{N}^k \mid m \leq n\}$.
Identify $\Omega^0_k := \mathbb{N}^k$ and set

$$s(m, n) = n$$
$$r(m, n) = m$$
$$d(m, n) = n - m$$
$$(\ell, m)(m, n) = (\ell, n)$$

Let $\mathbb{II}_k := \{(m, n) \in \Omega_k \mid m, n \leq 1_k\}$ and identify $\mathbb{II}^0_k := \{n \in \mathbb{N}^k \mid n \leq 1_k\}$.
(This example does not satisfy our standing assumption.)
The infinite path space $\Lambda^\infty$ and the path groupoid $G_\Lambda$

The *infinite path space* $\Lambda^\infty$ is the set of $k$-graph morphisms $x : \Omega_k \to \Lambda$. The topology on $\Lambda^\infty$ is generated by the cylinder sets

$$Z(\lambda) := \{ x \in \Lambda^\infty : \lambda = x(0, d(\lambda)) \} \quad \text{for } \lambda \in \Lambda.$$  

With this topology $\Lambda^\infty$ is a locally compact zero-dimensional space. For $q \in \mathbb{N}^k$ define the shift map: $\sigma^q : \Lambda^\infty \to \Lambda^\infty$ by

$$\sigma^q(x)(m, n) := x(m + q, n + q) \quad \text{for } x \in \Lambda^\infty \text{ and } (m, n) \in \Omega_k.$$  

Note $\sigma^q$ is a local homeomorphism.

We define the path groupoid $G_\Lambda \subset \Lambda^\infty \times \mathbb{Z}^k \times \Lambda^\infty$ by

$$G_\Lambda := \{(x, m - n, y) : \sigma^m(x) = \sigma^n(y) \text{ for some } m, n \in \mathbb{N}^k \}.$$  

The unit space is identified with $\Lambda^\infty$ via the map $x \mapsto (x, 0, x)$. Note $G_\Lambda$ is ample with topology generated by sets of the form

$$Z(\mu, \nu) := \{(x, m - n, y) : \sigma^m(x) = \sigma^n(y), \mu = x(0, m), \nu = y(0, n) \}.$$  

Cubes and Faces

Let $\Lambda$ be a $k$-graph. For $0 \leq n \leq k$ an element $\lambda \in \Lambda$ with

$$d(\lambda) = \varepsilon_{j_1} + \cdots + \varepsilon_{j_n} \quad \text{where} \quad j_1 < \cdots < j_n$$

is said to be an $n$-cube. Let $Q_n(\Lambda)$ denote the set of $n$-cubes. Note that 0-cubes are vertices and 1-cubes are edges.

For $n < 0$ or $n > k$, we have $Q_n(\Lambda) = \emptyset$.

With $\lambda \in Q_n(\Lambda)$ as above there is a unique quasimorphism $\varphi_\lambda : \mathbb{I}_n \to \Lambda$ compatible with the morphism $\iota : \mathbb{N}^n \to \mathbb{N}^k$ determined by $\iota(\varepsilon_i) = \varepsilon_{j_i}$ such that $\varphi_\lambda(1_n) = \lambda$.

We define the faces $F^0_i(\lambda), F^1_i(\lambda) \in Q_{n-1}(\Lambda)$, to be the unique $(n-1)$-cubes such that

$$\lambda = F^0_i(\lambda)\lambda_0 = \lambda_1 F^1_i(\lambda)$$

where $d(\lambda_\ell) = \varepsilon_{j_\ell}$ for $\ell = 0, 1$.

Fact: If $i < j$, then $F^\ell_i \circ F^m_j = F^m_{j-1} \circ F^\ell_i$.

The structure of cubes and face maps gives rise to a cubical set (see [Gr]).
Cubical homology

For \(1 \leq n \leq k\) define \(\partial_n : \mathbb{Z}Q_n(\Lambda) \to \mathbb{Z}Q_{n-1}(\Lambda)\) such that for \(\lambda \in Q_n(\Lambda)\)

\[
\partial_n(\lambda) = \sum_{i=1}^{n} \sum_{\ell=0}^{1} (-1)^{i+\ell} F^\ell_i(\lambda).
\]

It is straightforward to show that \(\partial_{n-1} \circ \partial_n = 0\).

Hence, \((\mathbb{Z}Q_*(\Lambda), \partial_*)\) is a complex and we define the homology of \(\Lambda\) by

\[H_n(\Lambda) = \ker \partial_n / \text{Im} \partial_{n+1}.
\]

The assignment \(\Lambda \mapsto H_*(\Lambda)\) is a covariant functor.

If \(\Lambda\) is a connected \(k\)-graph, \(H^0(\Lambda) = \mathbb{Z}\).

Example

Recall that \(C_m\) is a cycle with \(m\) vertices. We have

\[H_n(C_m) \cong \begin{cases} 
\mathbb{Z} & \text{if } n = 0, 1 \\
0 & \text{otherwise}.
\end{cases}
\]

If \(E\) is a connected 1-graph with finitely many vertices and edges, then \(H_1(E) \cong \mathbb{Z}^p\) where \(p = |E^1| - |E^0| + 1\) (i.e. the first Betti number of \(E\)).
The Künneth Theorem

Using basic homological algebra one can show:

**Theorem (Künneth Formula)**

Let $\Lambda_i$ be a $k_i$-graph for $i = 1, 2$. For $n \geq 0$ there is an exact sequence:

$$
0 \to \sum_{m_1+m_2=n} H_{m_1}(\Lambda_1) \otimes H_{m_2}(\Lambda_2) \xrightarrow{\alpha} H_n(\Lambda_1 \times \Lambda_2) \xrightarrow{\beta} \sum_{m_1+m_2=n-1} \text{Tor}(H_{m_1}(\Lambda_1), H_{m_2}(\Lambda_2)) \to 0.
$$

Let $\Lambda = C_2 \times C_1$ (see above).

By the Künneth Theorem we have

$$
H_0(\Lambda) \cong \mathbb{Z}, \quad H_1(\Lambda) \cong \mathbb{Z}^2, \quad H_2(\Lambda) \cong \mathbb{Z}.
$$
Acyclic $k$-graphs and free actions

A $k$-graph $\Lambda$ is said to be acyclic if $H_0(\Lambda) \cong \mathbb{Z}$ and $H_n(\Lambda) = 0$ for $n > 0$.

**Theorem**

Let $\Lambda$ be an acyclic $k$-graph and suppose that there is a free action of the group $G$ on $\Lambda$. Then there is an isomorphism:

$$H_*(\Lambda/G) \cong H_*(G).$$

**Example**

Let $\Lambda = \Delta_k := \{(p, q) \in \mathbb{Z}^k \times \mathbb{Z}^k \mid p \leq q\}$.

Identify $\Delta^0_k := \mathbb{Z}^k$ and use structure maps as given for $\Omega_k$.

Let $G = \mathbb{Z}^k$ act on $\Delta_k$ by translation.

It is easy to show that $\Delta_k$ is acyclic. We have $\Delta_k/\mathbb{Z}^k \cong T_k$ and so

$$H_n(T_k) \cong H_n(\mathbb{Z}^k) \cong \mathbb{Z}^k.$$
Topological realizations

One may construct the topological realization $X_\Lambda$ of a $k$-graph $\Lambda$ (see [KKQS]) by analogy with the geometric realization of a simplicial set.

Let $I = [0, 1]$. For $i = 1, \ldots, n$ and $\ell = 0, 1$ define $\eta^\ell_i : I^{n-1} \to I^n$ by

$$
\eta^\ell_i(x_1, \ldots, x_{n-1}) = (x_1, \ldots, x_{i-1}, \ell, x_i, \ldots, x_{n-1}).
$$

Then $X_\Lambda$, the topological realization of $\Lambda$, may be defined as the quotient of

$$
\bigsqcup_{n=0}^{k} Q_n(\Lambda) \times I^n
$$

by the equivalence relation generated by

$$
(\lambda, \eta^\ell_i(x)) \sim (F^\ell_i(\lambda), x)
$$

where $\lambda \in Q_n(\Lambda)$ and $x \in I^{n-1}$.

There is a natural isomorphism $H_*(\Lambda) \cong H_*(X_\Lambda)$. 
Cubical cohomology

Let $\Lambda$ be a $k$-graph and let $A$ be an abelian group. For $n \in \mathbb{N}$ set

$$C^n(\Lambda, A) = \text{Hom}(\mathbb{Z}Q_n(\Lambda), A)$$

and define

$$\delta^n : C^n(\Lambda, A) \to C^{n+1}(\Lambda, A) \text{ by } \delta^n(\varphi) = \varphi \circ \partial_{n+1}.$$

It is straightforward to show that $(C^*(\Lambda, A), \delta^*)$ is a complex.

We define the (cubical) cohomology of $\Lambda$ by

$$H^n(\Lambda, A) := Z^n(\Lambda, A)/B^n(\Lambda, A),$$

where $Z^n(\Lambda, A) := \ker \delta^n$ and $B^n(\Lambda, A) := \text{Im} \delta^{n-1}$.

Note $\Lambda \mapsto H^*(\Lambda, A)$ is a contravariant functor (covariant in $A$).

There is an alternative categorical cohomology theory that may be more useful in some circumstances.
The UCT and a long exact sequence.

Theorem (Universal Coefficient Theorem)

Let $\Lambda$ be a $k$-graph and let $A$ be an abelian group. Then for $n \geq 0$, there is a short exact sequence

$$0 \to \text{Ext}(H_{n-1}(\Lambda), A) \to H^n(\Lambda, A) \to \text{Hom}(H_n(\Lambda), A) \to 0.$$ 

By a standard argument, a short exact sequence of coefficient groups

$$0 \to A \to B \to C \to 0$$

gives rise to a long exact sequence

$$0 \to H^0(\Lambda, A) \to H^0(\Lambda, B) \to H^0(\Lambda, C) \to H^1(\Lambda, A) \to \cdots$$
$$\cdots \to H^{n-1}(\Lambda, C) \to H^n(\Lambda, A) \to H^n(\Lambda, B) \to H^n(\Lambda, C) \to \cdots$$
The $C^*$-algebra $C_\varphi(\Lambda)$

Suppose that $\Lambda$ is row-finite and source-free.

**Definition**

Let $\varphi \in Z^2(\Lambda, \mathbb{T})$. Let $C_\varphi^*(\Lambda)$ denote the universal $C^*$-algebra generated by a family of operators $\{t_\lambda : \lambda \in \Lambda^{\varepsilon_i}, 1 \leq i \leq k\}$ and a family of orthogonal projections $\{p_v : v \in \Lambda^0\}$ satisfying:

1. For $\lambda \in \Lambda^{\varepsilon_i}$, $t_\lambda^*t_\lambda = p_{s(\lambda)}$.
2. Suppose $\mu \nu = \nu' \mu'$ where $d(\mu) = d(\mu') = \varepsilon_i$, $d(\nu) = d(\nu') = \varepsilon_j$ and $i < j$. Then
   $$t_{\nu'}t_{\mu'} = \varphi(\mu \nu)t_{\mu}t_{\nu}.$$
3. For $v \in \Lambda^0$ and $i = 1, \ldots, k$,
   $$p_v = \sum_{\lambda \in v\Lambda^{\varepsilon_i}} t_\lambda t_\lambda^*.$$

If $\varphi$, $\varphi'$ are cohomologous, then $C_\varphi^*(\Lambda) \cong C_{\varphi'}^*(\Lambda)$.

There is an equivalent definition involving categorical cohomology.
Rotation algebras

Recall that $T_k = \mathbb{N}^k$.

There is precisely one 2-cube in $T_2$, namely $(1, 1)$.

Fix $\theta \in [0, 1)$. Let $\varphi \in Z^2(T_2, \mathbb{T})$ be given by $\varphi(1, 1) = e^{2\pi i \theta}$.

Then $C^*_\varphi(T_2)$ is the universal $C^*$-algebra generated by the unitaries $t_{\varepsilon_1}$ and $t_{\varepsilon_2}$ satisfying

$$t_{\varepsilon_2} t_{\varepsilon_1} = e^{2\pi i \theta} t_{\varepsilon_1} t_{\varepsilon_2}.$$

That is, $C^*_\varphi(T_2)$ is the rotation algebra $A_\theta$.

When $\theta = 0$, $C^*_\varphi(T_2) \cong C(\mathbb{T}^2)$.

If $\theta$ is irrational, $C^*_\varphi(T_2)$ is the well-known irrational rotation algebra.

More generally, every noncommutative torus arises as a twisted $k$-graph $C^*$-algebra $C^*_\varphi(T_k)$.

Crossed products of Cuntz algebras by quasifree actions may also be expressed as twisted 2-graph $C^*$-algebras (see below)
Categorical Cohomology

Let $\mathcal{C}$ be a small category and let $A$ be an abelian group.

The categorical cohomology, $H^*_{\text{cat}}(\mathcal{C}, A)$, is the usual cocycle cohomology for groupoids (see [R]) extended to small categories.

Denote the set of composable $n$-tuples in $\mathcal{C}$ by $\mathcal{C}^{(n)}$ where $\mathcal{C}^{(0)} = \mathcal{C}^0$.

Denote the set of functions $c : \mathcal{C}^{(n)} \to A$ by $C_{\text{cat}}^n(\mathcal{C}, A)$.

Define $\delta_{\text{cat}}^n : C_{\text{cat}}^n(\mathcal{C}, A) \to C_{\text{cat}}^{n+1}(\mathcal{C}, A)$ by

\[
(\delta_{\text{cat}}^n f)(\lambda_0, \ldots, \lambda_n) := f(\lambda_1, \ldots, \lambda_n)
+ \sum_{i=1}^{n} (-1)^i f(\lambda_0, \ldots, \lambda_{i-2}, \lambda_{i-1} \lambda_i, \lambda_{i+1}, \ldots, \lambda_n)
+ (-1)^{n+1} f(\lambda_0, \ldots, \lambda_{n-1}).
\]

Define $Z_{\text{cat}}^n(\mathcal{C}, A) := \ker \delta_{\text{cat}}^n$, $B^n(\mathcal{C}, A) := \text{Im } \delta_{\text{cat}}^{n-1}$ and

\[
H^n_{\text{cat}}(\mathcal{C}, A) := Z_{\text{cat}}^n(\mathcal{C}, A)/B^n_{\text{cat}}(\mathcal{C}, A).
\]
The $C^*$-algebra $C^*(\Lambda, c)$

Let $\Lambda$ be a $k$-graph and let $c$ be a normalized $\mathbb{T}$-valued categorical 2-cocycle.

**Definition**

Let $C^*(\Lambda, c)$ be the universal $C^*$-algebra generated by the family of operators $\{t_{\lambda} : \lambda \in \Lambda\}$ satisfying:

1. For $\lambda \in \Lambda$, $t_{s(\lambda)} = t^*_{\lambda}t_{\lambda}$;
2. For $\lambda, \mu \in \Lambda$, $t_{\lambda}t_{\mu} = c(\lambda, \mu)t_{\lambda\mu}$ if $s(\lambda) = r(\mu)$ and $t_{\lambda}t_{\mu} = 0$ otherwise;
3. For $v \in \Lambda^0, n \in \mathbb{N}^k$,

$$t_v = \sum_{\lambda \in v\Lambda^n} t_{\lambda}t^*_{\lambda}.$$  

It follows that $\{t_v : v \in \Lambda^0\}$ is a family of orthogonal projections.

**Remark:** If $c$ and $c'$ are cohomologous, then $C^*(\Lambda, c) \cong C^*(\Lambda, c')$.

**Theorem (see [KPS2])**

There is a map $c \in Z^2_{cat}(\Lambda, \mathbb{T}) \mapsto \sigma_c \in Z^2(\mathcal{G}_\Lambda, \mathbb{T})$ such that

$$C^*(\Lambda, c) \cong C^*(\mathcal{G}_\Lambda, \sigma_c).$$
Cohomology from a categorical perspective

Let $\mathcal{C}$ be a small category. A (right) $\mathcal{C}$-module is defined to be a contravariant functor from $\mathcal{C}$ to the category of abelian groups.

Given a module $A$, write $A_v$ for the abelian group assigned to the object $v$ and $A_\lambda : A_r(\lambda) \to A_s(\lambda)$ for the morphism assigned to $\lambda \in \mathcal{C}$.

A morphism between modules $\eta : A \to B$ is a natural transformation $\{\eta_v\}_v$.

For $A$ an abelian group, form the module $A^\mathcal{C}$ where $A^\mathcal{C}_v = A$ and $A^\mathcal{C}_\lambda = \text{id}$.

The category of $\mathcal{C}$-modules is an abelian category with sufficiently many projectives. Let $P_*$ be a projective resolution of $\mathbb{Z}^\mathcal{C}$.

The cohomology of $\mathcal{C}$ with coefficients in a $\mathcal{C}$-module $A$ is given by

$$\mathcal{H}^*(\mathcal{C}, A) = \text{H}^*(\text{Hom}_\mathcal{C}(P_*, A)).$$

Remark: If $A$ is an abelian group, $\mathcal{H}^*(\mathcal{C}, A^\mathcal{C}) \cong \text{H}^*_\text{cat}(\mathcal{C}, A)$. 
Isomorphism of the two cohomology theories

Let $\Lambda$ be a $k$-graph and let $A$ be an abelian group. By results of [KPS2]

$$H^n_{\text{cat}}(\Lambda, A) \cong H^n(\Lambda, A) \quad \text{for } n = 0, 1, 2.$$  

The methods used were ad hoc and no morphism $H^n_{\text{cat}}(\Lambda, A) \to H^n(\Lambda, A)$ was found for arbitrary $n$.

Gillaspy and Wu recently proved the isomorphism in general (see [GK]). They constructed an explicit projective resolution $\mathcal{P}^\text{cub}_*$ of $\mathbb{Z}\Lambda$ and an isomorphism of complexes:

$$\text{Hom}_\Lambda(\mathcal{P}^\text{cub}_*, A^\Lambda) \cong \text{Hom}(\mathbb{Z}Q_*(\Lambda), A).$$

The desired isomorphism then follows from

$$H^*_\text{cat}(\Lambda, A) \cong \mathcal{H}^*_*(\Lambda, A^\Lambda) = H^*_*(\text{Hom}_\Lambda(\mathcal{P}^\text{cub}_*, A^\Lambda))$$

$$\cong H^*_*(\text{Hom}(\mathbb{Z}Q_*(\Lambda), A))$$

$$= H^*_*(\Lambda, A).$$

They also give an explicit chain map $\nabla : \mathbb{Z}Q_n(\Lambda) \to \mathbb{Z}\Lambda^{(n)}$ that induces an isomorphism in homology:

$$\nabla(\lambda) := \sum_{\sigma \in \Sigma_n} (\text{sgn } \sigma)(\lambda_1^\sigma, \ldots, \lambda_n^\sigma) \quad \text{where } \lambda = \lambda_1^\sigma \cdots \lambda_n^\sigma, \ d(\lambda_i^\sigma) = \varepsilon_{\sigma(i)}.$$
From $\Lambda$-modules to equivariant sheaves

There is an exact functor from the category of $\Lambda$-modules to the category of $G_\Lambda$-sheaves (see [GK]).

Let $\mathcal{A}$ be a $\Lambda$-module and $x \in \Lambda^\infty$; so $x : \Omega_k \to \Lambda$ is a $k$-graph morphism. Then $\mathcal{A}_{x(p)}$ is an abelian group for each $p \in \mathbb{N}^k$ and $\mathcal{A}_{x(p,q)} : \mathcal{A}_{x(p)} \to \mathcal{A}_{x(q)}$ is a homomorphism for every $(p, q) \in \Omega_k$.

We obtain a directed system of abelian groups $(\mathcal{A}_{x(p)}, \mathcal{A}_{x(p,q)})$ and define

$$\mathcal{A}_x := \lim_{(p, q) \in \Omega_k} (\mathcal{A}_{x(p)}, \mathcal{A}_{x(p,q)}).$$

There is a natural topology on $\mathcal{A} := \bigsqcup \mathcal{A}_x$ making it into a sheaf over $\Lambda^\infty$. Let $\varphi^x_p : \mathcal{A}_{x(p)} \to \mathcal{A}_x$ be the natural map.

Action of $G_\Lambda$ on $\mathcal{A}$: Let $(x, n, y) \in G_\Lambda$ and let $a \in \mathcal{A}_y$. There are $p, q \in \mathbb{N}^k$ and $a_0 \in \mathcal{A}_{y(q)}$ such that $\sigma^p(x) = \sigma^q(y)$, $p - q = n$ and $a = \varphi^y_q(a_0)$. Then $x(p) = y(q)$ and so $a_0 \in \mathcal{A}_{x(p)}$. Define $(x, n, y) \cdot a := \varphi^x_p(a_0) \in \mathcal{A}_x$.

Note that $\mathcal{A} \mapsto \mathcal{A}$ is exact and so induces a natural map on cohomology:

$$\mathcal{H}^*(\Lambda, \mathcal{A}) \to H^*(G_\Lambda, \mathcal{A}).$$
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Thanks!
A **cubical set** is a triple $X = (X_n, \partial_i^\ell, f_i)$ consisting of sets $(X_n)^\infty_{n=0}$ and maps

$$
\partial_i^\ell : X_n \to X_{n-1} \quad 1 \leq i \leq n, \ \ell = 0, 1
$$

$$
f_i : X_{n-1} \to X_n \quad 1 \leq i \leq n
$$

satisfying the **cubical relations**

$$
\partial_i^\ell \partial_j^m = \partial_j^m \partial_{i+1}^\ell \quad \text{if } j \leq i,
$$

$$
f_i f_j = f_{i+1} f_j \quad \text{if } j \leq i,
$$

$$
\partial_i^\ell f_j = \begin{cases} 
  f_j \partial_i^{\ell-1} & \text{if } j < i, \\
  \text{id} & \text{if } j = i, \\
  f_{j-1} \partial_i^\ell & \text{if } j > i.
\end{cases}
$$

The maps $\partial_i^\ell$ are called *faces* and the $f_i$ are called *degeneracies*. (See [Gr].)

There is a natural cubical set associated with a higher rank graph.
Quasicubes in higher rank graphs

Let $n, k \in \mathbb{N}$. A monoid morphism $h : \mathbb{N}^n \to \mathbb{N}^k$ is *admissible* if there are $m \in \mathbb{N}$ and integers $1 \leq i_1 < \cdots < i_m \leq n$, $1 \leq j_1 < \cdots < j_m \leq k$ such that

$$h(\varepsilon_i) = \begin{cases} 
\varepsilon_{j_\ell} & \text{if } i = i_\ell \text{ for some } \ell, \\
0 & \text{otherwise}.
\end{cases}$$

An admissible map is said to be *degenerate* if $m < n$.

Fix a $k$-graph $\Lambda$.

A quasimorphism $\varphi : \mathbb{I}_n \to \Lambda$ is said to be an $n$-*quasicube*, if there is an admissible map $h : \mathbb{N}^n \to \mathbb{N}^k$ such that $h \circ d_{\mathbb{I}_n} = d_{\Lambda} \circ \varphi$.

Let $\tilde{Q}_n(\Lambda)$ denote the set of $n$-quasicubes.

Note that $n$-cubes give rise to $n$-quasicubes with nondegenerate degree maps.
The cubical set of a higher rank graph

Let $\Lambda$ be a higher rank graph.

For $\ell \in \{0, 1\}$, $1 \leq i \leq n$, there is a unique quasimorphism $\iota_i^\ell : \mathbb{I}_{n-1} \rightarrow \mathbb{I}_n$ such that

$$\iota_i^\ell(p) = (p_1, \ldots, p_{i-1}, \ell, p_i, \ldots, p_{n-1}) \quad \text{for } p \in \mathbb{I}^0_{n-1}.$$ 

Similarly, for $1 \leq i \leq n$, there is a unique quasimorphism $\vartheta_i : \mathbb{I}_n \rightarrow \mathbb{I}_{n-1}$ such that

$$\vartheta_i(p) := (p_1, \ldots, p_{i-1}, p_{i+1}, \ldots, p_n) \quad \text{for } p \in \mathbb{I}^0_n.$$ 

Observe that

$$\partial_i^\ell(\varphi) := \varphi \circ \iota_i^\ell \in \tilde{Q}_{n-1}(\Lambda) \quad \text{for } \varphi \in \tilde{Q}_n(\Lambda) \quad \text{and}$$

$$f_i(\varphi) := \varphi \circ \vartheta_i \in \tilde{Q}_n(\Lambda) \quad \text{for } \varphi \in \tilde{Q}_{n-1}(\Lambda).$$

One verifies that $(\tilde{Q}_n(\Lambda), \partial_i^\ell, f_i)$ is a cubical set.

The homology of $(\tilde{Q}_n(\Lambda), \partial_i^\ell, f_i)$ as defined in [Gr] coincides with $H_*(\Lambda)$. 
Crossed products of Cuntz algebras by quasifree actions

Let $\Lambda = B_2 \times C_1$ where $B_2$ is the 1-graph with one vertex and two edges. Note that $C^*(\Lambda) \cong O_2 \otimes C(\mathbb{T})$.

There are two 2-cubes in $\Lambda$, $a_j b$ for $j = 1, 2$. The boundary maps are trivial; so we have $Z^2(\Lambda, \mathbb{T}) = H^2(\Lambda, \mathbb{T}) \cong \mathbb{T}^2$ where $Z^2(\Lambda, \mathbb{T}) \ni \varphi \mapsto (\varphi(a_1 b), \varphi(a_2 b))$.

Fix $\varphi \in Z^2(\Lambda, \mathbb{T})$, say $\varphi(a_j b) = z_j$.

Then $C^*_\varphi(\Lambda)$ is isomorphic to the universal $C^*$-algebra generated by two isometries, $s_1, s_2$, and a unitary $u$ such that

$$s_1 s_1^* + s_2 s_2^* = 1 \quad \text{and} \quad us_j = z_j s_j u.$$

So $C^*_\varphi(\Lambda) \cong O_2 \rtimes_{\alpha} \mathbb{Z}$ where $\alpha(S_j) = z_j S_j$.

Moreover, every crossed product of $O_2$ by a quasifree automorphism is isomorphic to $C^*_\varphi(\Lambda)$ for some $\varphi$. 