The Brauer Group of a Groupoid

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The Brauer group $\text{Br}(G)$ of a locally compact groupoid $G$ is defined to be the set of Morita equivalence classes of actions of $G$ on elementary C*-bundles. Our main result is the isomorphism:

$$\text{Br}(G) \cong \text{Ext} (G, \mathbb{T}).$$

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Introduction

We define the Brauer group $\text{Br}(G)$ of a locally compact groupoid $G$ to be the set of Morita equivalence classes of pairs $(\mathcal{A}, \alpha)$ consisting of an elementary $C^*$-bundle $\mathcal{A}$ over $G^{(0)}$ (satisfing Fell’s condition) and an action $\alpha$ of $G$ on $\mathcal{A}$ by $*$-isomorphisms. The group operation is given by the tensor product.

If $G$ is a transformation groupoid, then $\text{Br}(G)$ is the equivariant Brauer group as defined in [CKRW].

If $G$ and $H$ are equivalent groupoids in the sense of [MRW], then $\text{Br}(G)$ and $\text{Br}(H)$ are isomorphic. It is shown that $\text{Br}(G)$ is isomorphic to $\text{Ext}(G, \mathbb{T})$, as defined by Renault.

If $G$ is étale, then $\text{Br}(G) \cong H^2(G, \mathcal{S})$, where $H^*(G, \cdot)$ is the natural extension of Grothendieck’s equivariant sheaf cohomology to étale groupoids. The assignment of such a cohomology class to a pair $(\mathcal{A}, \alpha)$ may be viewed as a generalized Dixmier-Douady invariant.
Continuous-trace $C^*$-algebras

A separable $C^*$-algebra $A$ is said to be of *continuous trace* if $X = \text{Prim} A$ is Hausdorff and there are ideals $I_1, I_2, \ldots$ such that each $I_n$ is Morita equivalent to an abelian $C^*$-algebra and such that the ideals $\{I_n\}_n$ generate $A$.

Dixmier and Doady constructed an invariant $\delta(A) \in H^2(X, \mathcal{S})$ (where $\mathcal{S}$ is the sheaf of germs of continuous $\mathbb{T}$-valued functions). They proved it has two interesting properties: First, $A$ is Morita equivalent to an abelian algebra iff $\delta(A) = 0$. Next, if $A_1$ and $A_2$ are continuous-trace $C^*$-algebras with $\text{Prim} A_1 = X = \text{Prim} A_2$ then $A = A_1 \otimes_{C_0(X)} A_2$ is also of continuous trace. Moreover, $\delta(A) = \delta(A_1) + \delta(A_2)$.

Green observed that two continuous-trace $C^*$-algebras $A_1$ and $A_2$, with $\text{Prim} A_1 = \text{Prim} A_2$, are Morita equivalent (in a spectrum preserving way) iff $\delta(A_1) = \delta(A_2)$. 
Elementary C*-bundles

It is natural to view a continuous-trace C*-algebra as arising from a continuous field of elementary C*-algebras (this was the point of view that Dixmier and Doady took). It is equivalent to view it as the section algebra of an elementary C*-bundle.

A C*-bundle $\mathcal{A}$ over a locally compact space $X$, write $p : \mathcal{A} \to X$, is said to be elementary if each fiber $\mathcal{A}_x$ is isomorphic to the algebra of compact operators on some complex Hilbert space and it satisfies Fell’s condition: for every $x_0 \in X$ there is an open neighborhood $U$ of $x_0$ and a continuous section $e$ of $\mathcal{A}$ over $U$ such that $e_x$ is a rank-1 projection for every $x \in U$.

Note that a C*-bundle $\mathcal{A}$ is elementary if and only if its section algebra $C_0(X; \mathcal{A})$ is a continuous-trace C*-algebra.
Reinventing the wheel

Let $\mathcal{A}$ and $\mathcal{B}$ be elementary C*-bundles over $X$. We may form the tensor product $\mathcal{A} \otimes \mathcal{B}$ (here $(\mathcal{A} \otimes \mathcal{B})_x = \mathcal{A}_x \otimes \mathcal{B}_x$). Then the associated section algebras behave as expected

$$C_0(X; \mathcal{A} \otimes \mathcal{B}) \cong C_0(X; \mathcal{A}) \otimes_{C_0(X)} C_0(X; \mathcal{B}).$$

Morita equivalence between the section algebras may also be recast in bundle theoretic terms in the obvious way. A slightly more elegant approach involves the linking algebra characterization of Brown, Green and Rieffel. We say that $\mathcal{A}$ and $\mathcal{B}$ are Morita equivalent if there is an elementary C*-bundle $\mathcal{C}$ containing $\mathcal{A}$ and $\mathcal{B}$ as complementary corners. This gives an equivalence relation on the class of elementary C*-bundles over $X$. Moreover, the tensor product operation respects this equivalence relation. It follows that Morita equivalence classes of elementary C*-bundles over $X$ form an abelian group under this operation.
The Brauer Group

The group that we obtain is called the Brauer Group $\text{Br}(X)$. The Dixmier-Doody invariant gives us an isomorphism

$$\text{Br}(X) \cong H^2(X, S).$$

This point of view is due to Green (unpublished seminar notes).

Given a locally compact group $G$ acting on $X$ one may consider the class of elementary C*-bundles over $X$ which are endowed with a compatible action of $G$. That is, we consider *-automorphic actions of $G$ on continuous-trace C*-algebras with spectrum identified with $X$ such that the induced action of $G$ on $X$ is the given one. The notion of Morita equivalence becomes more restrictive but one obtains a group under the tensor product operation.

This group is called the Equivariant Brauer Group (see [CKRW]). It can be quite interesting even if $X$ is a point.
Elementary $G$-C*-bundles: Definition

Now let $G$ be a locally compact groupoid and let $p : \mathcal{A} \to G^0$ be an elementary C*-bundle over its unit space: set

$$G * \mathcal{A} = \{(g, a) \mid s(g) = p(a)\}.$$ 

Then a continuous map $\alpha : G * \mathcal{A} \to \mathcal{A}$ is said to be a (left) action of $G$ on $\mathcal{A}$ if

i. $p(\alpha(g, a)) = r(g)$ for $(g, a) \in G * \mathcal{A},$

ii. $\alpha(gh, a) = \alpha(g, \alpha(h, a))$ for $(g, h) \in G^2$ and $(h, a) \in G * \mathcal{A}$

iii. $\alpha(x, a) = a$ for $x \in G^0$ and $p(a) = x.$

iv. the map $\alpha_g : A_{s(g)} \to A_{r(g)}$ is a *-isomorphism for $g \in G.$

We call the pair $(\mathcal{A}, \alpha)$ an elementary $G$-C*-bundle.
More on Elementary $G$-C*-bundles

A trivial example: Set $\mathcal{A} = G^0 \times \mathbb{C}$ and $\alpha_g = \text{id}$ for $g \in G$.

Let $(\mathcal{A}, \alpha)$ and $(\mathcal{B}, \beta)$ be elementary $G$-C*-bundles. We form the tensor product $\mathcal{A} \otimes \mathcal{B}$ as above and observe that there is a natural $G$-action $\alpha \otimes \beta$ given by

$$(\alpha \otimes \beta)_g(a \otimes b) = \alpha_g(a) \otimes \beta_g(b),$$

for $g \in G$, $a \in \mathcal{A}_{s(g)}$ and $b \in \mathcal{B}_{s(g)}$.

We say $(\mathcal{A}, \alpha)$ and $(\mathcal{B}, \beta)$ are Morita equivalent if there is an elementary $G$-C*-bundle $(\mathcal{C}, \gamma)$ such that $\mathcal{C}$ contains $\mathcal{A}$ and $\mathcal{B}$ as complementary corners and the restrictions of $\gamma$ to these corners gives $\alpha$ and $\beta$. 
The Brauer Group Revisited

As before Morita equivalence gives an equivalence relation on the class of elementary $G$-$C^*$-bundles. Moreover, the tensor product operation respects this equivalence relation. It follows that Morita equivalence classes of elementary $G$-$C^*$-bundles form an abelian semigroup $\text{Br}(G)$ under this operation.

Note that an elementary $G$-$C^*$-bundle $(\mathcal{A}, \alpha)$ is Morita equivalent to $(G^0 \times \mathbb{C}, \text{id})$ if there is a unitary action $u$ of $G$ on a Hilbert bundle $\mathcal{H}$ fibered over $G^0$ such that

$$(\mathcal{A}, \alpha) \cong (\mathcal{K}(\mathcal{H}), \text{Ad} \, u).$$

Fact: $\text{Br}(G)$ is an abelian group with identity element given by the class of $(G^0 \times \mathbb{C}, \text{id})$.

This group is called the Brauer group of $G$. 
The Group Case: Mackey Obstruction

We consider the case when $G^0$ is a singleton, that is, $G$ is a group. In this case an elementary $G$-$C^*$-bundle is given by $(\mathcal{K}, \alpha)$ where $\mathcal{K} = \mathcal{K}(H)$ is the algebra of compact operators on a Hilbert space $H$ and $\alpha : G \to \text{Aut}(\mathcal{K})$ is a continuous group homomorphism. It is well known, that every automorphism of $\mathcal{K}$ is of the form $\text{Ad} u$ for some unitary $u \in U(H)$. But it may not be possible to lift $\alpha$ to $U(H)$ (this happens iff $(\mathcal{K}, \alpha)$ is in the trivial class).

Note that for $u, v \in U(H)$, we have $\text{Ad} u = \text{Ad} v$ iff $u = tv$ for some $t \in \mathbb{T}$. We may form an extension $E_\alpha$ of $G$ by $\mathbb{T}$ which will be trivial exactly when $(\mathcal{K}, \alpha)$ is in the trivial class:

$$1 \to \mathbb{T} \to E_\alpha \to G \to 1$$

where $E_\alpha = \{(g, u) : \alpha_g = \text{Ad} u\}$. It is useful to think of $E_\alpha$ as the obstruction to lifting $\alpha$ to a unitary representation (cf. Mackey).
Group Extensions and Representations

Let \((\mathcal{K}, \alpha)\) be an elementary \(G\)-\(C^*\)-bundle. Given a central extension

\[
1 \to \mathbb{T} \xrightarrow{i} E \xrightarrow{\pi} G \to 1,
\]

then \(E \cong E_\alpha\) iff there is a unitary representation \(u : E \to U(H)\) such that for all \(e \in E\) and \(t \in \mathbb{T}\):

i. \(u(i(t)e) = tu(e)\)

ii. \(\alpha(\pi(e)) = \text{Ad} u(e)\)

It follows from this that two elementary \(G\)-\(C^*\)-bundles \((\mathcal{K}_1, \alpha_1)\) and \((\mathcal{K}_2, \alpha_2)\) are Morita equivalent iff \(E_{\alpha_1} \cong E_{\alpha_2}\).

Given an extension as in \((*)\), it is easy to construct a unitary representation \(u : E \to U(H)\) satisfying (i) above. One can then construct an elementary \(G\)-\(C^*\)-bundle \((\mathcal{K}, \alpha)\) using (ii). Then we have \(E \cong E_\alpha\).
The Brauer Group of a Group

It follows that the map from \((\mathcal{K}, \alpha) \mapsto E_\alpha\) induces a bijection from the set of Morita classes of elementary \(G\)-C*-bundles (that is, the set \(\text{Br}(G)\)) to isomorphism classes of central extensions of \(G\) by \(\mathbb{T}\) as in (*)..

Such extensions are classified by the abelian group \(H^2(G, \mathbb{T})\) (this is the cohomology of locally compact groups as defined by Moore). Thus we have a bijection \(\text{Br}(G) \rightarrow H^2(G, \mathbb{T})\).

Given two elementary \(G\)-C*-bundles \((\mathcal{K}_1, \alpha_1)\) and \((\mathcal{K}_2, \alpha_2)\) we have

\[
[E_{\alpha_1 \otimes \alpha_2}] = [E_{\alpha_1} \ast_{\mathbb{T}} E_{\alpha_2}] = [E_{\alpha_1}] + [E_{\alpha_2}],
\]

where \(E_{\alpha_1} \ast_{\mathbb{T}} E_{\alpha_2} = \{(e_1, e_2) \in E_{\alpha_1} \times E_{\alpha_1} : \pi_1(e_1) = \pi_2(e_2)\} / \sim\) and \((\iota_1(t)e_1, e_2) \sim (e_1, \iota_1(t)e_2)\). Hence, we have

\[
\text{Br}(G) \cong H^2(G, \mathbb{T}).
\]
Twists

We now return to the general case where $G$ is a locally compact groupoid equipped with a left Haar system. There is a natural analog of a central extension of a group by $\mathbb{T}$ called a \textit{twist}. A twist over $G$ is given by a pair of groupoid homomorphisms which identify unit spaces.

$$G^0 \times \mathbb{T} \overset{\iota}{\longrightarrow} E \overset{\pi}{\longrightarrow} G$$

where $\iota$ is injective, $\pi$ is surjective and for all $e \in E$ and $t \in \mathbb{T}$ we have $e\iota(s(e), t) = \iota(r(e), t)e$.

As before isomorphism classes of twists form an abelian group which we denote by $\text{Tw}(G)$. Define

$$[E_1] + [E_2] = [E_1 \star_{\mathbb{T}} E_2].$$

It is not generally the case that $\text{Tw}(G)$ and $\text{Br}(G)$ are isomorphic. But there is a map $\theta_G : \text{Tw}(G) \to \text{Br}(G)$. 

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The Map $\theta_G : \text{Tw}(G) \to \text{Br}(G)$

Consider the twist

$$G^0 \times \mathbb{T} \xrightarrow{\iota} E \xrightarrow{\pi} G.$$ 

By restricting to an appropriate summand of the left regular representation of $E$ we may find a unitary representation $u_E : E \to U(H_E)$ such that $u_E(\iota(r(e), t)e) = tu_E(e)$.

From this we construct an elementary $G$-$C^*$-bundles $(\mathcal{A}_E, \alpha_E)$ where $\mathcal{A}_E = G^0 \times \mathcal{K}(H_E)$ and $\alpha_E : \mathcal{K}(H_E) \to \mathcal{K}(H_E)$ is given by

$$(\alpha_E)_{\pi(e)} = \text{Ad}u_E(e).$$

The Morita class of $(\mathcal{A}_E, \alpha_E)$ only depends on $E$ and not on the choice of $u_E$. So we may define the map $\theta_G : \text{Tw}(G) \to \text{Br}(G)$ by $\theta_G([E]) = [(\mathcal{A}_E, \alpha_E)]$.

In general, $\theta_G$ is neither injective nor surjective.
**Range of $\theta_G$**

Define the natural map $\text{Br}(G) \to H^2(G^0, \mathcal{S})$ by $[(\mathcal{A}, \alpha)] \mapsto \delta(A)$ where $A = C_0(X; \mathcal{A})$ and let $\text{Br}_0(G)$ denote its kernel.

Fact: The range of $\theta_G$ is $\text{Br}_0(G)$.

Let $[(\mathcal{A}, \alpha)] \in \text{Br}_0(G)$. Then by stabilizing if necessary we may assume that $\mathcal{A} = G^0 \times \mathcal{K}(H)$ and so we obtain a continuous map $\alpha : G \to \text{Aut}(\mathcal{K}(H))$. We may now define a twist over $G$ as follows:

$$E_\alpha = \{(g, u) \in G \times U(H) : \alpha_g = \text{Ad} u\}.$$  

We observe that $\theta_G([E_\alpha]) = [(\mathcal{A}, \alpha)]$ and the fact is proven.


**Equivalence of Groupoids**

Muhly, Renault and Williams introduced the notion of equivalence of groupoids which mirrors Morita equivalence of C*-algebras.

Let $H$ and $G$ be locally compact groupoids, then a space $Z$ equipped with a left $H$-action and a right $G$-action is said to be an $H$-$G$ equivalence, if both actions are free and proper and the quotient map by each action may be identified with the fibering over the other’s unit space. Example, $G$ is a $G$-$G$ equivalence.

Given an elementary $G$-C*-bundle $(\mathcal{A}, \alpha)$ one may use $Z$ to get an elementary $H$-C*-bundle $(\mathcal{A}^Z, \alpha^Z)$ as follows. Set

$$\mathcal{A}^Z = \{(z, a) \in Z \times \mathcal{A} : s(z) = p(a)\}/\sim$$

where $(z \cdot g, a) \sim (z, \alpha_g(a))$; and $\alpha^Z_h(z, a) = (h \cdot z.a)$. This map induces an isomorphism $\text{Br}(G) \cong \text{Br}(H)$. 


Equivalence of twists

Let $Z$ be an $H$-$G$ equivalence and let $F$ and $E$ be twists over $H$ and $G$ respectively. We say that $F$ and $E$ are $Z$-equivalent if there is an $F$-$E$ equivalence compatible with $Z$.

Now let $F$ and $E$ be twists over $G$, then $\theta_G([F]) = \theta_G([E])$ iff they are $G$-equivalent. Hence, $\ker \theta_G$ consists of twists $G$-equivalent to the trivial twist. Such twists arise from principal $\mathbb{T}$-bundles over $G^0$. Set $\mathcal{E}(G) = \text{Tw}(G)/\ker \theta_G$. Then $\theta_G$ induces an isomorphism

$$\mathcal{E}(G) \cong \text{Br}_0(G').$$

By expanding our horizons to include twists over groupoids equivalent to $G$ (and organizing them somehow into a group) one obtains a group isomorphic to $\text{Br}(G)$. 

Main Result

Let $X$ be a space and let $\psi : X \rightarrow G^0$ be a local homeomorphism, then the groupoid

$$G^\psi = X \ast G \ast X = \{(x, g, y) : \psi(x) = r(g), s(g) = \psi(y)\}$$

is equivalent to $G$ by the equivalence $X \ast G$. There is a pull-back map $\psi^* : \text{Tw}(G) \rightarrow \text{Tw}(G^\psi)$ which induces an injection $\psi^* : \mathcal{E}(G) \rightarrow \mathcal{E}(G^\psi)$; these maps are compatible with composition. So we can take the inductive limit. Set

$$\text{Ext} \ (G, \mathbb{T}) = \lim_{\psi} \mathcal{E}(G^\psi).$$

Main Theorem: Let $G$ be a locally compact group with left Haar system. Then

$$\text{Br}(G) \cong \text{Ext} \ (G, \mathbb{T}).$$
Selected References


