Math 285  Section 1003  Spring 2018

EXAM I

Name (in CAPITALS):  

Signature:  

Date:  

READ AND FOLLOW THESE INSTRUCTIONS

• This booklet contains 11 pages, including this cover page. Be sure none are missing.
• Print all the requested information above and sign your name.
• This is a closed-booked exam. Only scientific calculators are allowed.
• There are 10 problems.
• After you finish the exam, turn in the entire booklet.

You MUST show all the work in order to receive full credit.
Unsupported answers will receive little credit.
1. (6 points) State the order of
\[
\frac{d^3y}{dx^3} + \left( \frac{d^2y}{dx^2} \right)^3 - 5y \frac{dy}{dx} = 4xy
\]
and state whether the equation is linear or nonlinear. If the equation is nonlinear, then circle all nonlinear terms in the equation.

3rd order -
nonlinear b/c \( \left( \frac{d^2y}{dx^2} \right)^3 \) & \( 5y \frac{dy}{dx} \)

2. (10 points) Given: \( \frac{dy}{dx} = xy^2 \).

Solve by separation of variables. Write the solution in explicit form.

\[
\int \frac{dy}{y^2} = \int x \, dx \implies \frac{y^{-1}}{-1} = \frac{x^2}{2} + C
\]

\[
- \frac{1}{y} = \frac{x^2 + 2C}{2}
\]

\[
y = \frac{-2}{x^2 + 2C} \quad (0,0)
\]

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3. (16 points) Solve the IVP by integrating factor: \( xy' + 4y = x^2, y(1) = 2 \). Write the solution in explicit form. Give the largest interval \( I \) over which this solution is defined.

\[
\int P(x) \, dx = \int \frac{4}{x} \, dx = 4 \ln|x| = x^4
\]

\[
y' + \frac{4}{x} y = x^4
\]

\[
\int \frac{d}{dx} (x^4 y) \, dx = \int x^5 \, dx
\]

\[
x^4 y = \frac{x^6}{6} + C
\]

\[
y = \frac{x^2}{6} + C x^{-4} \quad (10 \text{ points})
\]

\[
y(1) = 2
\]

\[
2 = \frac{1^2}{6} + C 1^{-4} \quad \Rightarrow \quad 2 - \frac{1}{6} = C
\]

\[
C = \frac{11}{6}
\]

\[
y = \frac{x^2}{6} + \frac{11}{6} x^{-4} \quad (3 \text{ points})
\]
4. (12 points) Consider the autonomous differential equation: \[ \frac{dy}{dx} = 2y - y^2. \]

(a) Find all critical points and the phase portrait.

(b) Classify each critical point as asymptotically stable, unstable, or semi-stable.

(c) State the equilibrium solutions.

(d) Sketch the equilibrium solutions and typical solution curves in the region of the xy-plane determined by the graphs of the equilibrium solutions.

\[
\begin{align*}
a) & \quad 2y - y^2 = 0 \\
& \quad y(2-y) = 0 \\
& \quad y = 0, \quad y = 2 \\

\text{and} \\
\begin{align*}
b) & \quad \frac{dy}{dx} < 0 \\
& \quad \frac{dy}{dx} > 0 \\
& \quad \frac{dy}{dx} < 0 \\
& \quad \text{stable} \\
& \quad \text{unstable} \\
\end{align*}
\end{align*}
\]
5. (13 points) Consider the homogeneous differential equation: \( x^2y'' - 5xy' + 9y = 0 \).

Given that \( y_1 = x^3 \) is one solution to this homogeneous equation, find the general solution using reduction of order.

I. way

\[
y_2 = u(x) \cdot x^3
\]

\[
y_2' = u' \cdot x^3 + u \cdot 3x^2
\]

\[
y_2'' = u'' \cdot x^3 + u' \cdot 3x^2 + u' \cdot 3x^2 + u \cdot 6x
\]

\[
x^2 \left[ u'' \cdot x^3 + 6x^2 u' + 6x^3 u \right] - 5x \left[ u' \cdot x^3 + 3u x^2 \right] + 9 \left[ u \cdot x^3 \right] = 0
\]

\[
x^2 u'' + u' \left[ 6x^4 - 5x^4 \right] + u \left[ 6x^3 - 15x^3 + 9x^3 \right] = 0
\]

\[
x'' \cdot x^5 + u' \cdot x^4 = 0
\]

Let \( v = u' \) & \( v' = u'' \).

\[
v' \cdot x^5 + v \cdot x^4 = 0
\]

So. V.

\[
\frac{dv}{dx} = -v \cdot \frac{x^4}{x^5} \quad \Rightarrow \quad \int \frac{dv}{v} = \int -\frac{dx}{x}
\]

\[
\ln |v| = -\ln |x|
\]

\[
\ln |v| = \ln |x|^{-1}
\]

\[
v = \frac{1}{x}
\]

\[
v = \int \frac{du}{dx} \cdot \frac{1}{x} \cdot dx \quad \Rightarrow \quad u = \ln |x|
\]

\[
\Rightarrow \quad y_2 = \ln |x| \cdot x^3
\]

\[
\Rightarrow \quad y = c_1 x^3 + c_2 x^3 \ln |x|\]
\[ y_2(x) = y_1(x) \int \frac{e^{-\int P(x) \, dx}}{y_1^2(x)} \, dx \]

**Standard form:** \[ y'' - \frac{5}{x} y' + \frac{9}{x^2} y = 0 \quad P(x) = -\frac{5}{x} \]
\[ y_1(x) = x^3 \]
\[ y_1^2(x) = (x^3)^2 = x^6 \]

\[ y_2(x) = x^3 \int \frac{e^{\int \frac{5}{x} \ln |x| \, dx}}{x^6} \, dx = x^3 \int \frac{e^{\ln |x|^5}}{x^6} \, dx \]

\[ = x^3 \int \frac{dx}{x} = x^2 \cdot \ln |x| \quad (10 \text{ pts}) \]

\[ \Rightarrow y = c_1 x^3 + c_2 x^3 \cdot \ln |x| \quad (3 \text{ pts}) \]
6. (10 points) \( y_1(x) = e^{4x} \) and \( y_2(x) = e^{-6x} \) are two solutions to \( y'' + 2y' - 24y = 0 \) on the interval \((-\infty, \infty)\).

(a) Determine whether \( \{ y_1, y_2 \} \) is a linearly independent set of functions on \((-\infty, \infty)\) by computing the Wronskian of \( y_1 \) and \( y_2 \).

(b) Explain why \( \{ y_1, y_2 \} \) is a fundamental set of solutions on this ODE on \((-\infty, \infty)\).

(c) Write the general solution to the ODE on \((-\infty, \infty)\).

\[
\text{a) } W(y_1, y_2) = \begin{vmatrix}
4e^{4x} & e^{-6x} \\
4e^{4x} & -6e^{-6x}
\end{vmatrix} = -6e^{2x} - 4ie^{-2x}
\]

\[
= -10e^{-2x} \neq 0 \quad \forall x \in (-\infty, \infty)
\]

Since \( W(y_1, y_2) \neq 0 \) on \((-\infty, \infty)\), then \( y_1 \) and \( y_2 \) are linearly independent.

b) \( y_1 \) & \( y_2 \) are linearly independent solutions of

\[ \text{ODE } y'' + 2y' - 24y = 0 \text{, that's why they form fundamental set of solutions.} \]

c) \[ y = c_1y_1 + c_2y_2 = c_1e^{4x} + c_2e^{-6x} \]
7. (10 points) The population of a country doubles in 40 years. If the rate of growth is proportional to the number of individuals present at any time, determine the time necessary for the individuals to triple.

\[
\begin{align*}
\frac{dx}{dt} &= kx, \\
x(t_0) &= x_0.
\end{align*}
\]

Solution of this IVP is \(x(t) = x_0 e^{kt}\).

\[
\begin{align*}
X(t) &= X_0 e^{kt} \\
2X_0 &= X_0 e^{k \cdot 40} \\
2 &= e^{40k} \\
\ln 2 &= 40k \\
k &= \frac{\ln 2}{40} \approx 0.01733.
\end{align*}
\]

\[
X(t) = X_0 e^{0.01733t}.
\]

\[
\begin{align*}
3X_0 &= X_0 e^{0.01733t} \\
\ln 3 &= 0.01733t \\
t &= \frac{\ln 3}{0.01733} \approx 63.4 \text{ years}
\end{align*}
\]

\[
k = \frac{\ln 2}{40} \quad \text{and} \quad t = \frac{\ln 3}{\ln 2} \quad \text{are also ok!}
\]
8. (10 points) Solve following homogeneous differential equation: \( y'' + 25y = 0. \)

\[
m^2 + 25 = 0 \quad m^2 = -25
\]

\[
m = \pm 5i \quad x = 0
\]

\[
y = c_1 e^{0x} \cos(5x) + c_2 e^{0x} \sin(5x)
\]

\[
y = c_1 \cos(5x) + c_2 \sin(5x)
\]
9. (13 points) Find the general solution of the following differential equation

\[ R_0 \frac{dq}{dt} + \frac{1}{C_0} q = E_0, \text{ (series circuits)} \]

where \( R_0, C_0 \) and \( E_0 \) are constants.

\[
\begin{align*}
\frac{dq}{dt} + \frac{1}{R_0 C_0} q &= \frac{E_0}{R_0} \\
\int \frac{1}{R_0 C_0} dt &= \int \frac{E_0}{R_0} dt \\
\frac{1}{R_0 C_0} t &= \frac{E_0}{R_0} t + C \\
q &= \frac{E_0}{R_0} t + C \\
\end{align*}
\]

The correct simplification is:\n
\[ q = \frac{E_0}{C_0} + C e^{-\frac{t}{R_0 C_0}} \]
10. (5 points) Extra Credit - Verify that \( u = \frac{c_1 e^x}{1 + c_1 e^x} \) is a solution of the differential equation \( \frac{du}{dx} = u(1 - u) \).

\[
\begin{align*}
  u &= \frac{c_1 e^x}{1 + c_1 e^x} \\
  u' &= \frac{c_1 e^x (l + c_1 e^x) - c_1 e^x \cdot c_1 e^x}{(l + c_1 e^x)^2} \\
  &= \frac{c_1 e^x x + c_1 e^{2x} x - c_1 e^{2x}}{(1 + c_1 e^x)^2} \\
  \frac{du}{dx} &= u(1 - u) \\
  \frac{du}{dx} &= \frac{c_1 e^x}{(1 + c_1 e^x)^2} \\
  u(1 - u) &= \frac{c_1 e^x}{1 + c_1 e^x} \left( 1 - \frac{c_1 e^x}{1 + c_1 e^x} \right) \\
  &= \frac{c_1 e^x}{1 + c_1 e^x} \cdot \frac{l + c_1 e^x - c_1 e^x}{1 + c_1 e^x} = \frac{c_1 e^x}{(1 + c_1 e^x)^2}
\end{align*}
\]
This blank page is for your solutions. Remember to turn in this page!