1 Overview

My research interests are in algebraic topology, especially rational homotopy theory and its interactions with combinatorial and geometric group theory and algebraic geometry. Algebraic topology is a branch of mathematics using the techniques of abstract algebra to study the property of topological spaces. The tools of algebraic topology have been broadly used in many fields in pure mathematics, mathematical physics as well as recently in data analysis.

Rational homotopy theory was founded by Quillen and Sullivan in late 1960’s, studying classification of the rational homotopy types of spaces depending on the rational homotopy groups. This simplifies the homotopy groups to vector spaces in a remarkably computational way. In particular, the rational homotopy types of formal spaces are classified by the rational cohomology algebras. Rational homotopy theory contributes fruitful results in topology, geometry and algebra. For example, there is a close relationship between classifying rational homotopy types and purely algebraic deformation theory of commutative algebras. Schlessinger and Stasheff [26] investigated the classification of rational homotopy types by filtered models (constructed by Halperin and Stasheff [17]) via algebraic deformation theory. In another direction, Kadeishvili [19] studied transferred “homotopy commutative” on cohomology and classified them using the deformation of Harrison cochain complex.

The theory of cohomology jump loci is a popular research field in algebraic topology using the techniques in algebraic geometry, especially algebraic varieties and schemes, to study the properties of topological spaces. The tools of cohomology jump loci have been significantly used in hyperplane arrangements, singularity theory and rational homotopy theory. For example, the tangent cone theorem by Dimca, Papadima and Suciu [15] provides an important obstruction to formality in rational homotopy theory, complementary to higher Massey products. The degree-one cohomology jump loci depending only on the fundamental group reveal important properties in group theory.

Applying these techniques to several interesting spaces and groups in geometric topology and group theory, e.g., arrangements, configuration spaces, braid groups, link groups, etc., will broaden the scope of the research, and finding positive evidence for developing various theoretical aspects.

In the following sections, I will provide an overview of my current research projects and my future research directions for the next five years. Most of the applications on concrete examples of spaces and groups could be done in collaborations with graduate students and senior undergraduate students.

1.1 Algebraic models in rational homotopy theory. In my Ph.D. work with Suciu [29, 30], we investigated several properties of commutative differential graded algebra (CDGA) models and various Lie algebras of spaces and groups (especially their formality properties), and the way such properties behave under products, coproducts, split injections and change of coefficient fields.

Fix a connected commutative graded algebra $H$. Motivated by the work of Schlessinger and Stasheff [26], in my current project with Rogers [25], we construct a direct and explicit $L_\infty$-quasi-isomorphism between the following two (filtered) differential graded Lie algebras (DGLAs): the derivations of the Halperin and Stasheff’s bigraded model of $H$ and the Harrison cochain complex of $H$. A relation between the above two DGLAs was studied by Block and Lazarev [7]. The key feature of our approach is that the $L_\infty$-quasi-isomorphism is functorial (up to unique homotopy) and depends only on $H$. Furthermore, passing to the Deligne-Getzler-Hinich $\infty$-groupoids, we produce a homotopy equivalence of the corresponding simplicial sets. In particular, on the level of $\pi_0$, we obtain an explicit bijection from the moduli space of filtered models with cohomology $H$ to the moduli space of $C_\infty$-structures on $H$. Our goal is to explicitly describe the $C_\infty$-structures on group cohomology. In our future work, we will investigate the geometric properties of these moduli spaces as homotopy schemes over DG schemes.
I will also explore formulas computing $C_\infty$-structures on group cohomology in lower dimensions, and their relationship with Massey products, and some other algebraic invariants, e.g., topology complexity, depth, the Lusternik-Schnirelmann category, and the cohomology jump loci.

1.2 Cohomology jump loci. Cohomology jump loci include characteristic varieties studying the moduli space of rank 1 local systems on a space and their ‘linear’ approximation: resonance varieties. The dimensions of the irreducible components of the first resonance variety relate the lower central series (LCS) ranks of the maximal metabelian quotient of the fundamental group. This is known as Chen ranks formula, conjectured by Suciu for arrangement groups and recently proved by Cohen and Schenck [12] for a broad class of groups. In my paper [27] with Suciu, we explored several properties of resonance varieties and Chen ranks, especially their behavior respect to products and coproducts. In my recent paper [31] with Suciu, based on previous work by several authors, we refined a method for computing the first resonance variety, the first resonance scheme and the Chen ranks using the Gröbner basis of the infinitesimal Alexander invariant.

In my current project [32] with Suciu, we further investigate the Chen ranks formula to refine the result and make it fit a broader class of groups, by considering resonance varieties of CDGA models. We will investigate a generalized tangent cone theorem relating the characteristic varieties of spaces and resonance varieties of CDGAs. Another future research direction is to explore the resonance varieties of the transferred $C_\infty$-structures on the cohomology via deformation theory. Following the work of Papadima and Suciu [24], I also plan to explore the relationship between the cohomology jump loci and the Bieri-Neumann-Strebel invariants.

1.3 Braid groups, arrangements, configuration spaces, etc. In my work [29, 30] with Suciu, we have answered several questions in rational homotopy theory and group theory regarding one-relator groups, nilpotent groups, link groups, and fundamental groups of Seifert fibered manifolds.

In my paper [27] with Suciu, we investigated the resonance varieties and detected the formality properties of the pure virtual braid groups, which are also known as the triangular groups corresponding to the Yang-Baxter equation studied by Bartholdi, Enriquez, Etingof and Rains [3]. In my survey paper [28] with Suciu, we explored various algebraic invariants from combinatorial and geometric group theory of braid groups, virtual braid groups and welded braid groups.

Cohen, Pakianathan, Vershinin, and Wu [13] computed the cohomology algebras of the upper pure welded braid groups $wP_n^+$ and asked a question: Are the pure braid group $P_n$ and $wP_n^+$ isomorphic for $n \geq 4$? In my recent paper [31] with Suciu, we computed the resonance variety, the resonance scheme and the Chen ranks of $wP_n^+$, and as an application, we answered their question. By comparing the corresponding resonance varieties, we also answered a split injection question by Bellingeri.

For every quiver of finite type, there is a finitely presented group called a picture group, which was introduced recently by Igusa, Orr, Todorov, and Weyman. I am working on a project in [33] studying the rational homotopy theory of the picture groups of type $A_n$, especially the various algebraic invariants including the CDGA models, LCS ranks, Chen ranks and the cohomology jump loci.

Let $E$ be an elliptic curve. An elliptic arrangement is a finite collection of fibers of group homomorphisms $E^{\times n} \to E$. In the project [6] with Bibby and Pagaria, we are working to determine when the complement of an elliptic arrangement is formal.

Most of the application projects in §5 are suitable for collaborating with graduate students and senior undergraduate students. Several classes of spaces and groups, e.g., arrangements, Artin groups, braid groups, picture groups, etc., also relate interesting properties and formulas in combinatorics and graph theory. I also program using mathematical software such as GAP, Macaulay2, Mathematica and SageMath. Students completing such a project will be benefit in mathematics and programming skills.
2 Moduli spaces in rational homotopy theory

2.1 Halperin and Stasheff’s filtered models. Fix a simply-connected graded commutative algebra $H$ over a field $k$ of characteristic zero. A classical question in algebraic topology is to describe all the rational homotopy types with cohomology $H$. An equivalent purely algebraic question is to describe all CDGAs up to weak equivalence with cohomology $H$. In [17], Halperin and Stasheff constructed the bigraded model $(\bigwedge Z, d)$ of $H$, which is in analog with Tate–Józefiak resolution in algebra. By perturbing the differential of the bigraded model, they constructed a filtered model $(\bigwedge Z + \rho d)$ for each CDGA with cohomology $H$, and this filtered model is unique up to CDGA isomorphism. The advantage is that this provides a method to use all perturbations (only of the differential) to describe all CDGAs and all the rational homotopy types with cohomology $H$. The key advantage is that

Example 1. ([17]) let $H$ be the cohomology algebra $H^*(S^2 \vee S^2 \vee S^3; k)$, i.e., $H$ has a basis $x_1$, $x_2$ and $x_3$ of degree 2, 2 and 5 with zero products. The bigraded model is given by $(\bigwedge Z, d)$, where $Z$ is a vector space with basis $x_1$, $x_2$, $x_3$, $x_11$, $x_12$, $x_22$, $x_112$, $x_122$, ..., and the differential $d$ is given by $d(x_i) = 0$, $d(x_{ij}) = x_i x_j$, $d(x_{ijk}) = x_i x_{jk} - x_k x_{ij}$, ... Up to isomorphism there are exactly two classes of filtered models, one with $\rho = 0$ and the other one with nonzero $\rho(x_{ijk}) = a_i x_3$. Hence, up to weak equivalence, there are two classes of CDGA with cohomology $H$. This shows that there are exactly two rational homotopy types with cohomology $H$.

2.2 Homotopy Transfer Theorem. A $C_\infty$-structure on a graded $k$-vector space $V$ is a family of graded $k$-multilinear maps

$$\mu_n : V^\otimes n \to V, \text{ for } n \geq 1,$$

of degree $2 - n$ satisfying the Stasheff identities, and each of the operations $\mu_n$ annihilates the images of shuffle products. As shown in the Homotopy Transfer Theorem by Kadeishvili in [19], any CDGA $A$ over $k$ gives a $C_\infty$-structure on its cohomology $H$, and the transferred $C_\infty$-structure is unique up to $C_\infty$-isomorphism. This gives another method to use all $C_\infty$-structures on $H$ to describe the rational homotopy types with cohomology $H$. The advantage here is that $H$ is smaller than the bigraded model.

Example 1. (continue) ([19]) For the same algebra $H = H^*(S^2 \vee S^2 \vee S^3; k)$ in example 1, there are exactly two isomorphic classes of minimal $C_\infty$-structures on $H$. One with all trivial products $\mu_n$, while the other one with nontrivial products $\mu_3(x_1, x_1, x_2) = a_1 x_3$ and $\mu_3(x_1, x_2, x_2) = a_2 x_3$. This example reveals the explicit relationship between the filtered model and the transferred $C_\infty$-structure.

2.3 Moduli spaces. In [26], Schlessinger and Stasheff studied the classification of rational homotopy type using algebraic deformation theory. The filtered models with cohomology $H$ are “controlled” by the DGLA consisting all weight decreasing derivations of $(\bigwedge Z, d)$. On the other hand, as in [19], all $C_\infty$-structures on $H$ are “controlled” by the DGLA, Harrison cochain complex of $H$. Using the bridge of rational homotopy type or an argument of cobar-bar resolution we can see an implicit bijection corresponding between these two moduli spaces. Although a quasi-isomorphism between the derivation DGLA of the bigraded model of $H$ and the Harrison cochain complex of $H$ is known in [7], there is no explicit map between them sending $\rho$ to $\mu_i$ as revealed in Example 1. Our goal is to construct such an explicit map in the following project.

Project 1 (with C. Rogers [25]). Construct an explicit and direct bijection from the moduli space of filtered models with cohomology $H$ to the moduli space of $C_\infty$-structures on $H$.

To achieve the goal of this project, we construct an explicit $L_\infty$-quasi-isomorphism between the following two (filtered) DGLAs: the derivations of the bigraded model of $H$, and the Harrison cochain
complex of $H$. Taking their Deligne-Getzler-Hinich $\infty$-groupoids, we produce a homotopy equivalence of the corresponding simplicial sets. In particular, on the level of $\pi_0$, we obtain a bijection from the moduli space of filtered models with cohomology $H$ to the moduli space of $C^\infty$-structures on $H$. In our future work, we will investigate the geometric properties of these moduli spaces as homotopy stacks over DG schemes.

3 Formality

A commutative differential graded algebra (CDGA) is formal, if it is quasi-isomorphic to its cohomology algebra with zero differential. A connected space $X$ is said to be formal if its CDGA model is formal, i.e., the cohomology algebra gives a CDGA model for $X$. If there is a CDGA morphism from the $i$-minimal model to the cohomology algebra $H^*(X;\mathbb{Q})$ induces isomorphisms in cohomology up to degree $i$ and a monomorphism in degree $i+1$, then $X$ is called $i$-formal. The 1-formality of $X$ depends only on the fundamental group $G = \pi_1(X)$. We find it useful to separate the 1-formality property of a group $G$ into two complementary properties: filtered-formality and graded-formality.

$$\text{formal} \iff i\text{-formal} \iff 1\text{-formal} \iff \text{graded-formal} + \text{filtered-formal}.$$}

3.1 Lie algebras and algebraic models. There is a close relationship between CDGA models and Lie algebras of the fundamental group $G = \pi_1(X)$. The Lie algebra dual to the first stage of the minimal model is isomorphic to the Malcev Lie algebra $m(G;\mathbb{Q})$. The Chevalley–Eilenberg complex of the Malcev Lie algebra gives the first stage of the minimal model. By a theorem of Quillen, the associated graded Lie algebra of $m(G;\mathbb{Q})$ is isomorphic to $\text{gr}(G;\mathbb{Q})$, which is the graded Lie algebra associated to the lower central series filtration of $G$.

The holonomy Lie algebra $h(G;\mathbb{Q})$ is a quadratic approximation of $\text{gr}(G;\mathbb{Q})$, constructed from the cup product map in low degrees. In paper [29] with Suciu, we give an explicit formula for those cup products, and an algorithm for computing the holonomy Lie algebra, using a Magnus expansion method. The computation of the Malcev Lie algebra and the graded Lie algebra of the group $G$ is much more difficult. In many contexts, it is of much interest to find presentations for the Lie algebras corresponding to some specific classes of groups, for instance, pure braid-like groups, fundamental groups of configuration spaces, and picture groups.

3.2 Formality properties. A finitely generated group $G$ is called filtered-formal, if the Malcev Lie algebra $m(G;\mathbb{Q})$ is isomorphic to the degree completion of the graded Lie algebra $\text{gr}(G;\mathbb{Q})$. The group $G$ is called graded-formal, if $\text{gr}(G;\mathbb{Q})$ is isomorphic to the holonomy Lie algebra $h(G;\mathbb{Q})$.

We prove that the 1-formality, graded-formality and filtered-formality notions have the following “propagation” properties, which play an important role in detecting partial formality properties of groups, for instance, in the proof of Theorem 6 below.

**Theorem 1** ([29]). (1) If $G$ is 1-formal and the injection $N \to G$ is split, then $N$ is 1-formal.
(2) The free product $G_1 \ast G_2$ is 1-formal $\iff$ the product $G_1 \times G_2$ is 1-formal $\iff$ $G_1$ and $G_2$ are 1-formal.
(3) Filtered formality and graded formality also satisfy these “propagation” properties.

We also provide a close connection between filtered formality of a group and homogeneous weights on the minimal model.

**Theorem 2** ([29]). A finitely generated group $G$ is filtered-formal if and only if the canonical 1-minimal model of $G$ is filtered-isomorphic to a 1-minimal model with positive Hirsch weights.
This theorem and work of Morgan [21] imply that fundamental groups of complex smooth algebraic varieties are filtered-formal. Furthermore, we showed in [29] that the fundamental group of an orientable Seifert manifold is always filtered-formal, but not 1-formal if the Euler number is not zero.

3.3 Descent of formality properties. As shown by Sullivan, Halperin–Stasheff, and Neisendorfer–Miller, the formality of a path-connected space with finite Betti numbers is independent of the field \( k \) of characteristic 0. This result is very useful to determine the formality of some smooth manifolds using the de Rham model, e.g., any compact Kähler manifold is a formal space, as proved by Deligne, Griffiths, Morgan, and Sullivan. We proved that partial formality properties are also independent of the field \( k \) of characteristic 0. More precisely, let \( X \) be a path-connected space with finite Betti numbers.

**Theorem 3** ([29]). (1) The space \( X \) is \( i \)-formal over \( \mathbb{Q} \) if and only if \( X \) is \( i \)-formal over \( k \).
(2) The group \( G = \pi_1(X) \) is filtered-formal (graded-formal) over \( \mathbb{Q} \) if and only if \( G \) is filtered-formal (graded-formal) over \( k \).

Applying this descent theorem to filtered formality of finitely generated torsion-free nilpotent groups recovers a theorem of Cornulier in [14] for finitely generated nilpotent Lie algebras.

One direction to generalize these formality results is to consider more general coefficient rings. This closely relates to pro-\( p \) groups and Massey products in Galois cohomology, see, e.g., [18].

**Project 2.** Investigate CDGA models and formality properties of spaces and groups when \( k \) is a field of characteristic \( p \).

4 Cohomology jump loci

4.1 Resonance varieties and characteristic varieties. The resonance varieties of a space \( X \), which originated from the study of complements of hyperplane arrangements by Falk, are subvarieties of \( H^1(X; \mathbb{C}) \). More generally, the resonance varieties of a complex CDGA \( (A^*, d) \) with finite dimensional \( A^1 \), recently studied by Dimca, Papadima, Suciu, and others, are defined by

\[
\mathcal{R}_k(A, d) := \{ a \in A^1 \mid \dim(H^1(A; \delta_a)) \geq k \}, \quad \text{where } \delta_a(u) = d(u) + a \cdot u \text{ for } u \in A^1.
\]

The resonance varieties of a connected CW-complex \( X \) of finite type are the resonance varieties of \( H^*(X; \mathbb{C}) \) with zero differential. The characteristic varieties of \( X \) are the jumping loci for cohomology with coefficients in rank 1 local systems,

\[
\mathcal{V}_k^i(X) := \{ \rho \in \text{Hom}(\pi_1(X), \mathbb{C}^*) \mid \dim(H^i(X; \mathbb{C}_\rho)) \geq k \}.
\]

The degree-1 cohomology jump loci depend only on the fundamental group \( G = \pi_1(X) \). In this case, as shown in [15], these two classes of varieties are closely related by the Tangent Cone Theorem: If \( G \) is 1-formal, then \( \text{TC}_1(\mathcal{V}_k^1(G)) = \mathbb{R}_k^1(G) \), and all irreducible components of \( \mathbb{R}_k^1(G) \) are rationally defined linear subspaces of \( H^1(G; \mathbb{C}) \). This yields new and powerful obstructions for a finitely generated group \( G \) to be 1-formal.

**Project 3** (with A. Suciu). Prove the Tangent Cone Theorem for filtered-formal groups using the resonance varieties of finite CDGA models of such groups, i.e., \( \text{TC}_1(\mathcal{V}_k^1(G)) = \mathbb{R}_k^1(A(G), d) \) for a filtered-formal group \( G \) with a finite CDGA model \( A(G) \). Apply the result to obtain obstructions to filtered formality.
The Alexander invariants, originating from the study of the Alexander polynomials of knots and links, play an important role in investigating resonance varieties, characteristic varieties and Chen ranks. One approach to Project 3 is generalizing the Alexander invariants of spaces and groups to CDGA models. Along with Project 3, I will also investigate the relations between the resonance varieties, the characteristics varieties, and the Bieri-Neumann-Strebel invariants of finitely generated groups.

4.2 Chen ranks and resonance varieties. K.T. Chen studied the lower central series (LCS) quotients of the maximal metabelian quotient \( G/G'' \), of a group \( G \), which were called Chen groups by Murasugi, who used them to study the Milnor invariants of links. The LCS ranks of the Chen groups are called the Chen ranks, and are denoted by \( \theta_k(G) \). Cohen and Suciu developed Massey’s method and found the Chen ranks of the pure braid groups by computing the Gröbner basis of the Alexander invariants. Generalizing a theorem in [22], we connect the Chen Lie algebra and the associated graded Lie algebra of a filtered-formal group.

**Theorem 4** ([29]). If \( G \) is a filtered-formal group, then the Chen Lie algebra \( \text{gr}(G/G''; k) \) is isomorphic to \( \text{gr}(G; k)/\text{gr}(G; k)'' \).

There is a close relationship between Chen ranks and resonance varieties. Suciu conjectured that the Chen ranks of an arrangement group \( G \) are given by

\[
\theta_k(G) = \sum_{m \geq 2} c_m \cdot \theta_k(F_m), \quad \text{for } k \gg 0,
\]

where \( c_m \) is the number of \( m \)-dimensional components of \( R_1^1(G) \), and \( F_m \) is the free group with \( m \) generators. Recently, D. Cohen and Schenck [12] showed that, for a finitely presented, commutator-relators 1-formal group \( G \), the Chen ranks formula holds, provided the components of \( R_1^1(G) \) are zero-isotropic, projectively disjoint, and reduced as schemes. In [32], we further investigate the Chen ranks formula.

**Project 4** (with A. Suciu). Prove a generalized Chen ranks formula for filtered-formal groups, using the resonance varieties of CDGA models.

4.3 Cohomology jump loci and \( C_\infty \)-algebras. The resonance varieties are first defined on the cohomology algebra of a space \( X \), i.e., a graded commutative algebra, and have been well studied in the past twenty years. They have been generalized to study CDGA in the recent 3 or 4 years, and fruitful results appear in this field. In a very recent paper [8], Budur and Rubió have been started studying the cohomology jump loci using deformation theory. This reveals the intrinsic relationship between Project 1 and Projects 3 and 4, which provides opportunity for completing these projects and exploring further new projects.

5 Spaces and groups

Braid groups have showed their importance in several important fields of mathematics. The classical Artin braid groups \( B_n \) are subgroups of the automorphism groups of free groups \( \text{Aut}(F_n) \), containing the pure braid groups \( P_n = \text{IA}_n \cap B_n \). Several interesting new families of braid-like groups have gained importance over the past twenty years: pure welded braid groups \( wP_n \), pure virtual braid groups \( vP_n \), and pure braid groups on Riemann surfaces \( P_{g,n} \), which fit the following diagram.
5.1 Pure welded braid groups. The welded braid groups $wB_n$ are subgroups of $\text{Aut}(F_n)$, also named as braid-permutation group by Fenn, Rimányi, and Rourke. The pure welded braid groups, $wP_n = wB_n \cap IA_n$, are also known as McCool groups or motion groups. There is a class of interesting subgroups of $wP_n$, denoted by $wP_n^+$. F. Cohen, Pakianathan, Vershchin, and Wu [13] computed the cohomology algebras of $wP_n^+$ and asked a question: Are $P_n$ and $wP_n^+$ isomorphic for $n \geq 4$? We answered this question by investigating the resonance varieties and Chen ranks of $wP_n^+$, by computing the Gröbner basis of the infinitesimal Alexander invariants of $wP_n^+$.

Theorem 5 ([31]). The resonance varieties $R^1(wP_n^+)$ are given by a union of $(j+1)$-dimensional linear subspaces $L_{i,j}$ of $\mathbb{C}^{(i)}$ for $1 \leq j < i \leq n - 1$. The Chen ranks are given by $\theta_1 = \binom{n}{2}$, $\theta_2 = \binom{n}{3}$, and $\theta_k = \sum_{i=3}^{k} \binom{n+i-2}{i+1} + \binom{n+1}{4}$ for $k \geq 3$.

Question: Whether or not the resonance varieties of $wP_n^+$ equal the complements the Bieri-Neumann-Strebel invariants of $wP_n^+$?

5.2 Pure virtual braid groups. The virtual braid groups come from the theory of virtual knots introduced by L. Kauffman. A presentation of the pure virtual braid group $vP_n$ was given by Bardakov. The groups $vP_n$ and their subgroups $vP_n^+$ were also independently studied by Bartholdi, Enriquez, Etingof and Rains [3] and Lee [20], as groups arising from the Yang-Baxter equations. Using the Tangent Cone Theorem and the propagation properties of 1-formality, we proved the following result, which was announced in [3].

Theorem 6 ([27]). The pure virtual braid groups $vP_n$ and $vP_n^+$ are 1-formal if and only if $n \leq 3$.

The pure virtual braid groups provide examples which are graded-formal but not filtered-formal. The cohomology algebras, resonance varieties, LCS ranks, Chen ranks, formality and Koszulness of these braid-like groups have been investigated by several groups of researchers over the last ten years. These results are summarized in our survey paper [28]. The Vassiliev invariants of these braid-like groups have been studied recently, see, e.g., paper of Bar-Natan and Dancso [1].

Problem: Compute the Bieri-Neumann-Strebel invariants of $vP_n^+$ and $vP_n$.

The three types of braid crossings mentioned above are depicted in Figure 1.

5.3 Pure braid groups on Riemann surfaces. Another important class of braid-like groups are the pure braid groups on compact Riemann surfaces $\Sigma_g$ of genus $g$, denoted by $P_{g,n}$. Much work has been done for this class of groups, see [2, 4, 10, 16], but still there are several unsolved problems.

Project 5. Compute the resonance varieties and the Bieri-Neumann-Strebel invariants of $P_{g,n}$, the resonance varieties of the CDGA model of $P_{g,n}$, and the Chen ranks of $P_{g,n}$ and explore their relationship.
The groups $P_{g,n}$ are always filtered-formal, but not 1-formal for $g = 1$ and $n \geq 3$. We have computed several examples for $P_{g,n}$, which allow us to conjecture some results for this project, and these conjectures show positive evidence for Project 3 and Project 4.

5.4 Configuration spaces and arrangements. The configuration space of $n$ ordered points in a connected manifold $M$ is defined to be

$$\text{Conf}_n(M) := \{(x_1, \ldots, x_n) \in M^n \mid x_i \neq x_j \text{ for } i \neq j\}.$$ 

The configuration space $\text{Conf}_n(\mathbb{C})$ is a classifying space for the pure braid group $P_n$, while the configuration space $\text{Conf}_n(\Sigma_g)$ is a classifying space for $P_{g,n}$. Algebraic models of $\text{Conf}_n(M)$ were studied by F. Cohen and Taylor for $M = \mathbb{R}^l$, and by Fulton and MacPherson, Kříž, and Totaro for $M$ a smooth projective variety. Lambrechts and Stanley recently studied the configuration spaces of a simply-connected manifold and made several interesting conjectures.

Graphic configuration spaces are generalizations of classical configuration spaces. Lima-Filho and Schenck gave a LCS ranks formula for graphic arrangements. Bibby and Hilburn [5] studied algebraic models of certain graphic configuration spaces when the graph is chordal. The Malcev Lie algebras of graphic configuration spaces on Riemann surfaces were determined by Berceanu, et al. [4]. These results and conjectures provide interesting directions for research.

There is a natural action of symmetric group $S_n$ on the configuration space $\text{Conf}_n(M)$ by permutation of coordinates. Church and Farb [11] introduced representation stability to study certain sequences of groups and spaces, e.g., pure braid-like groups and configuration spaces.

**Project 6.** Explore the representation stability of CDGA models of (graphic) configuration spaces and the representation stability of resonance varieties of these algebraic models. Compute the Malcev Lie algebras and Chen ranks using these algebraic models.

Let $E$ be an elliptic curve. An elliptic arrangement is a finite collection of fibers of group homomorphisms $E^\times \to E$.

**Project 7** (with Bibby and Pagaria [6]). Determine when an elliptic arrangement is formal.

In [4], the formality question have been answered for graphic configuration spaces on Riemann surfaces of genus $g$, in particular for for elliptic braid arrangements when $g = 1$. We are working on this project to answer the formality question for any elliptic arrangements.

5.5 Picture groups from quiver representations. For every quiver of finite type, there is a finitely presented group called a picture group, which was introduced recently by Igusa, Orr, Todorov, and Weyman. They proved that the integral cohomology groups of the picture group $G(A_n)$ of type $A_n$ with straight orientation are free abelian with ranks given by the ‘ballot numbers’. As shown by Igusa, the classifying space of the category of non-crossing partitions is a $K(G(A_n), 1)$.
We noticed that for each picture group $G(A_n)$ there is a right-angled Artin group $R(A_n)$ such that these two groups have the same resonance varieties. Hence, the resonance varieties for these groups can be determined from the results of Papadima and Suciu [23].

**Project 8.** Construct a finite CDGA model for $G(A_n)$ and compute the Malcev Lie algebra of $G(A_n)$. Investigate the relationship between the LCS ranks, the Chen ranks of $G(A_n)$ and the resonance varieties of the CDGA model. Explore properties of picture groups from these algebraic invariants.

We conjectured a finite CDGA model for $G(A_n)$. Using this model, we conjectured that $G(A_n)$ is filtered-formal, but not 1-formal by computing non-trivial Massey products. Picture groups provide examples for Project 3 and Project 4.

### 5.6 Research projects for students.
Most of the application projects in §5 on concrete spaces and groups can be done in collaboration with graduate students and senior undergraduate students with prerequisites of algebraic topology, group theory, commutative algebra and homological algebra.

Several classes of spaces and groups, e.g., hyperplane arrangements, Artin groups, pure braid groups, etc., also relate interesting combinatorics and graph theory properties and formulas. The Hilbert series, the LCS ranks and the Chen ranks of these groups also contains very interesting combinatorial information.

For example, the Betti numbers in the sequences of pure braid-like groups in §5.1 and §5.2 are famous combinatorial numbers: the Betti numbers of $P_n$ and $wP_n^+$ are the Stirling numbers of the first kind, while the Betti numbers of $vP_n^+$ are the Stirling numbers of the second kind and the Betti numbers of $vP_n$ are the Lah numbers. In §5.5, the Betti numbers of the picture group of type $A_n$ relates Ballot numbers and Catalan numbers. The LCS ranks and Chen ranks of right-angled Artin groups involves clique polynomials and cut polynomials. There are several interesting projects to explore the combinatorics of the Betti numbers, LCS ranks and Chen ranks of sequences of groups in §5.

We also need to program by mathematical software such as GAP, Macaulay2, Mathematica and SageMath to complete these projects in §5. These program problems are also suitable for senior undergraduate and graduate students research projects. I have written several packages and scripts to carry out computations for my papers and current research projects in this section. Completing such a project will benefit the students in many ways in mathematics and programming skills.

### References:


