1. Area under a curve.

2. Definite Integral
Let \( f(x) \) be a continuous function defined on the interval \([a, b]\). The definite integral (accumulated change) of \( f(x) \) from \( a \) to \( b \) is

\[
\int_a^b f(x) \, dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i) \Delta x.
\]

3. The relation between area under a curve and the definite integral

\[
\int_a^b f(x) \, dx = \text{(The Area above x-axis)} - \text{(The Area under x-axis)}
\]

4. More properties definite integral

\[
\int_a^b f(x) \pm g(x) \, dx = \int_a^b f(x) \, dx \pm \int_a^b g(x) \, dx
\]

\[
\int_a^b kf(x) \, dx = k \int_a^b f(x) \, dx
\]

\[
\int_a^b f(x) \, dx = \int_a^c f(x) \, dx + \int_c^b f(x) \, dx
\]
MATH 1231 Worksheet on Area Approximation with Rectangles

1a. Use 2 left rectangles to approximate the area under the curve \( f(x) = 10 - x^2 \) and above the interval \([0, 2] \). Sketch the rectangles using the graph of \( f(x) \) below, and find \( L_2 \), the sum of their areas. 

\[
\Delta x = \frac{2-0}{2} = 1
\]

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>10</td>
</tr>
<tr>
<td>1</td>
<td>9</td>
</tr>
</tbody>
</table>

\( L_2 = (10 + 9) \Delta x = 19 \)

1b. Use 2 right rectangles to approximate the area under the curve \( f(x) = 10 - x^2 \) and above the interval \([0, 2] \). Sketch the rectangles using the graph of \( f(x) \) below, and find \( R_2 \), the sum of their areas.

\[
\Delta x = \frac{2-0}{2} = 1
\]

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<tr>
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<tbody>
<tr>
<td>1</td>
<td>9</td>
</tr>
<tr>
<td>2</td>
<td>6</td>
</tr>
</tbody>
</table>

\( R_2 = (9 + 6) \Delta x = 15 \)

1c. Use 2 midpoint rectangles to approximate the same area as in (1a) and (1b). Sketch the rectangles using the graph of \( f(x) \) below, and find \( M_2 \), the sum of their areas. 

\( M_2 = 17.5 \)
\[ \Delta x = \frac{2 - 0}{2} = 1 \]

\[ M_2 = (9.75 + 7.75) \Delta x = 17.5 \]

2a. Use 4 left rectangles to approximate the same area as in problem 1. Find the sum \( L_4 \).
\[ L_4 = (10 + 9.75 + 9 + 7.75) \Delta x = 18.25 \]

2b. Use 4 right rectangles to approximate the same area as in problem 1. Find the sum \( R_4 \).
\[ R_4 = (9.75 + 9 + 7.75 + 6) \Delta x = 16.25 \]
2c. Use 4 midpoint rectangles to approximate the same area as in problem 1. Show that \( M_4 = 17.375 \) square units. Then find the value of the following quantity: \( \frac{1}{3} \left( \frac{L_4 + R_4}{2} \right) + \frac{2}{3}M_4 = 17.33 \). This quantity is a very accurate approximation of the area.

\[ \Delta x = \frac{2 - 0}{4} = 0.5 \]

\[ \begin{array}{c|c}
 x & f(x) \\
 0.25 & 9.9375 \\
 0.75 & 9.4375 \\
 1.25 & 8.4375 \\
 1.75 & 6.9375 \\
\end{array} \]

\[ M_4 = (9.9375 + 9.4375 + 8.4375 + 6.9375) \cdot 0.5 = 17.375 \]

3a. A portion of the graph of \( y = x^2 - 16x + 72 \) is given below. Make a careful sketch of the region whose area is given by the definite integral: \( \int_2^{14} (x^2 - 16x + 72) \, dx \). Then shade the region.

3b. Estimate the area (in square units) of the region in part (a) using the 4 right rectangle approximation.

\[ \Delta x = \frac{14 - 2}{4} = 3 \]

\[ \begin{array}{c|c}
 x & f(x) \\
 5 & 17 \\
 8 & 17 \\
 11 & 17 \\
 14 & 44 \\
\end{array} \]

\[ R_4 = \left( 17 + 8 + 17 + 44 \right) \Delta x \]

\[ = 86 \times 3 \]

\[ = 258 \]
In problems 1–5, find an antiderivative of the given function.

1. \( g(x) = 6x^2 - 4x^6 \)  
   \[ G(x) = 2x^3 - \frac{4}{7}x^7 \]

2. \( f(x) = 3e^x - \frac{1}{x} + 12 \)  
   \[ F(x) = 3e^x - \ln|x| + 12x \]

3. \( h(x) = 3^x - x^3 \)  
   \[ H(x) = \frac{3^x}{\ln 3} - \frac{x^4}{4} \]

4. \( m(x) = 12x^{-3} + 6\sqrt{x} - 100 \)  
   \[ M(x) = -6x^{-2} + 4\sqrt[3]{4x} - 100x \]

5. \( r(x) = 13(1.05)^x - x \)  
   \[ R(x) = 13 \left( \frac{1.05^x}{\ln 1.05} \right) - \frac{x^2}{2} \]

In problems 6–11, evaluate the given integral, i.e., find the indicated general antiderivative.

6. \[ \int (x + 1) \, dx = \int x^2 + x \, dx = \frac{x^3}{3} + \frac{x^2}{2} + C \]

7. \[ \int \left( \frac{x + 2}{x} \right) \, dx = \int 1 + \frac{2}{x} \, dx = x + 2\ln|x| + C \]

8. \[ \int (x + 1)^3 \, dx = \frac{(x+1)^4}{4} + C \]

9. \[ \int d(\ln(x+1)) \, dx = \ln(x+1) + C \]

10. \[ \frac{d}{dx} \left( \int \ln(x+1) \, dx \right) = \ln(x+1) \]

11. \[ \int \left( \frac{1}{2\sqrt{x^4} + \sqrt{x^3}} \right) \, dx = \int \frac{1}{x^{\frac{4}{3}}} + \frac{2}{x^{\frac{7}{3}}} \, dx = -\frac{3}{2}x^{-\frac{1}{3}} + \frac{2}{5}x^{\frac{5}{3}} + C \]

12. \[ \int \frac{1}{2x + 1} \, dx = \frac{\ln|2x+1|}{2} + C \]