1. The definition of Limits

- The limits of \( f(x) \) when \( x \) approach \( a \) is denoted by
  \[
  \lim_{x \to a} f(x)
  \]

- \( x \to a \) means that \( x \) approach \( a \), i.e. \( x \) is close to \( a \) very much. \((x \neq a)\)

Example 1:

\[
\lim_{x \to 2} (x^2 + 1) = 5.
\]

Example 2:

\[
\lim_{x \to 1} \left( \frac{x^2 - 1}{x - 1} \right) = \lim_{x \to 1} \left( \frac{(x + 1)(x - 1)}{x - 1} \right) = \lim_{x \to 1} (x + 1) = 2.
\]

When the function \( f(x) \) is continuous and \( f(a) \) is well-defined, then

\[
\lim_{x \to a} f(x) = f(a),
\]

denominator is NOT zero.

2. Derivative at a point by Limits

The derivative of \( f(x) \) at \( a \) is given by

\[
f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}
\]

Example 3: (Q1 is Homework2 in textbook section 2.4 page 163)
Q1. Estimate the derivative of function \( f(x) = -x^2 + 4x \) at \( x = 3 \), to the nearest tenth. Calculator work in the last page.
Q2. Using limit definition compute \( f'(3) \).

<table>
<thead>
<tr>
<th>( x \to 3 )</th>
<th>( f(x) )</th>
<th>( \frac{f(x)-f(3)}{x-3} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.99</td>
<td>3.019</td>
<td>-1.99</td>
</tr>
<tr>
<td>2.999</td>
<td>3.001999</td>
<td>-1.999</td>
</tr>
<tr>
<td>3.01</td>
<td>2.9799</td>
<td>-2.01</td>
</tr>
<tr>
<td>3.001</td>
<td>2.997999</td>
<td>-2.001</td>
</tr>
</tbody>
</table>

\[ f(3) \approx -2.0 \]
3. Differentiability of a function
- Points with no derivatives.

- A function \( f(x) \) is **differentiable** at \( a \) if the derivative \( f'(a) \) exists.
- A function \( f(x) \) is **differentiable** over an open interval if the derivative \( f'(x) \) exists for all points in the interval.

4. Derivative of a function by Limits

![Derivative of function](image)

**The derivative of function \( f(x) \) is given by**

\[
 f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}
\]

\[
 \frac{\Delta f(x)}{\Delta x} \approx \frac{f(x)}{x} \bigg|_{x=0}
\]

**Example 4:** Let \( f(x) = 2x^2 \).

(a) Find the average rate of change of \( f(x) \) between the points \((x, f(x))\) and \((x+h, f(x+h))\)

\[
\text{ARC} = \frac{f(x+h) - f(x)}{x+h - x} = \frac{2x^2 + 4xh + 4h^2 - 2x^2}{h} = \frac{4xh + 4h^2}{h} = 4x + 4h
\]

(b) Find \( f'(x) \) using the limit definition and the answer (a)

\[
 f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{(2x^2 + 4xh + 4h^2) - 2x^2}{h} = 4x
\]
Example 5: Let \( f(x) = 1 - 2x - 3x^2 \).
(a) Find the average rate of change of \( f(x) \) between the points \((x, f(x))\) and \((x+h, f(x+h))\)

\[
\text{ARC} = \frac{f(x+h) - f(x)}{x+h - x} = -2h - 6x - 3h^2
\]

\[
= \frac{1 - 2(x+h) - 3(x+h)^2 - (1 - 2x - 3x^2)}{h} = -2 - 6x - 3h
\]

\[
= \frac{1 - 2x - 2h - 3x^2 - 6xh - 3h^2 - 1 + 2x + 3x^2}{h} = -2 - 6x - 3h
\]

(b) Find \( f'(x) \) using the limit definition and the answer (a)

\[
f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} (-2 - 6x - 3h) = -2 - 6x
\]

Example 6: Let \( f(x) = 5 - 3x + 4x^2 \).
(a) Find the average rate of change of \( f(x) \) between the points \((x, f(x))\) and \((x+h, f(x+h))\)

\[
\text{ARC} = \frac{f(x+h) - f(x)}{x+h - x} = \frac{-3h + 8x + 4h^2}{h}
\]

\[
= \frac{5 - 3(x+h) + 4(x+h)^2 - (5 - 3x + 4x^2)}{h} = -3 + 8x + 4h
\]

\[
= \frac{5 - 3x - 3h + 4x^2 + 8xh + 4h^2 - 5 + 3x - 4x^2}{h} = -3 + 8x + 4h
\]

(b) Find \( f'(x) \) using the limit definition and the answer (a)

\[
f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} (-3 + 8x + 4h) = -3 + 8x
\]
Calculator work for Example Q1: (HW2 in textbook section 2.4, page 163)
Estimate the derivative of function \( f(x) = -x^2 + 4x \) at \( x = 3 \), to the nearest tenth.

1. Press "Y=".
2. After \( Y1=\), put in \(-X^2 + 4X\)
3. After \( Y2=\), put in \((Y1(X) - Y1(3))/(X - 3)\)
   
   Press (-) for the negative sign.
   
   Press "X,T,\theta,n" for \( X \).
   
   Press VARS, \( \blacktriangleright \), Enter, Enter for \( Y1 \).

4. Go back to Screen. Press "2ND" then Press "MODE"(QUIT).
5. Put in \( Y2(2.99) \), then Enter.
   
   Press VARS, \( \blacktriangleright \), scroll down, then Enter, Enter for \( Y2 \).

Exercise: Let \( f(x) = -x^2 + 4x \).
(a) Find the average rate of change of \( f(x) \) between the points \( (x, f(x)) \) and \( (x + h, f(x + h)) \)
(b) Find \( f'(x) \) using the limit definition and the answer (a)
(c) Verify the result above

\[
ARC = \frac{f(x+h) - f(x)}{x+h-x}
\]

\[
= \frac{-(x+h)^2+4(x+h)-(x^2+4x)}{h}
\]

\[
= \frac{\text{[Redo]} - 2x-2x-h^2+4x+4h + x^2+4x}{h}
\]

\[
= \frac{-2xh-h^2+4h}{h} = -2x-h+4
\]

\[
f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}
\]

\[
= \lim_{h \to 0} (-2x-h+4)
\]

\[
= -2x+4
\]