To receive full credit for a problem you must show all necessary work.

1. (10 points) Use the arc length formula to find the length of the curve

\[ y = 3(2x - 1)^{3/2} \]

for \(1 \leq x \leq 3\).

\[ y' = \frac{9}{2} (2x-1)^{1/2} 2 = 9(2x-1)^{1/2} \]

By arc length formula

\[ L = \int_{1}^{3} \sqrt{1 + (y')^2} \, dx = \int_{1}^{3} \sqrt{1 + 81(2x-1)} \, dx \]

\[ = \int_{1}^{3} \sqrt{162x - 80} \, dx \]

\[ = \left[ \frac{406}{162} \right]_{82}^{406} U^{3/2} \, du \]

\[ = \left. \frac{1}{243} \left( 406^{2/3} - 82^{2/3} \right) \right| \]

\[ = \frac{1}{243} \left( 406^{2/3} - 82^{2/3} \right) \]
2. (10 points) Calculate the center of mass (centroid) of a lamina with density $\rho = 6$ and shape $R$ bounded by $y = 2x$ and the parabola $y = x^2$.

- The area of the region $R$ is

$$A = \int_0^2 2x - x^2 \, dx = x^2 - \frac{x^3}{3} \bigg|_0^2 = \frac{4}{3}$$

- The $x$-coordinate of the centroid $\bar{x}$ is

$$\bar{x} = \frac{1}{A} \int_0^2 x(f(x) - g(x)) \, dx$$

$$= \frac{3}{4} \int_0^2 x(2x - x^3) \, dx$$

$$= \frac{3}{4} \left[ \frac{2}{3}x^3 - \frac{x^4}{4} \right]_0^2$$

$$= \frac{3}{4} \left( \frac{2}{3} \cdot 2^3 - \frac{2^4}{4} \right) = \frac{3}{4} \left( \frac{8}{3} - \frac{8}{4} \right) = 1$$

- The $y$-coordinate of the centroid $\bar{y}$ is

$$\bar{y} = \frac{1}{2A} \int_0^2 f(x)^2 - g(x)^2 \, dx$$

$$= \frac{1}{2} \cdot \frac{3}{4} \int_0^2 4x^2 - x^4 \, dx$$

$$= \frac{3}{8} \left( \frac{4}{3}x^3 - \frac{x^5}{5} \right) \bigg|_0^2 = \frac{8}{5}$$

So, the centroid of the lamina is $(1, \frac{8}{5})$. 

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3. (1) (6 points) For what values of \( r \) does the function \( y = e^{rx} \) a solution for \( y'' - 4y = 0 \)?

\[
y' = re^{rx} \\
y'' = r^2e^{rx}
\]

plug into the equation

\[
r^2e^{rx} - 4e^{rx} = 0
\]

\[
(r^2 - 4)e^{rx} = 0
\]

since \( e^{rx} \neq 0 \), we have

\[
r^2 - 4 = 0
\]

\[
(r + 2)(r - 2) = 0
\]

\[
r = 2 \quad \text{and} \quad r = -2
\]

so \( y = e^{2x} \) and \( y = e^{-2x} \) are solutions for \( y'' - 4y = 0 \).

(2) (4 points) If \( r_1 \) and \( r_2 \) are the values of \( r \) that you found in part (1), show that every member of the family of functions \( y = ae^{r_1x} + be^{r_2x} \) is a solution for \( y'' - 4y = 0 \).

\[
y = ae^{r_1x} + be^{r_2x}
\]

\[
y' = 2ae^{r_1x} + 2be^{r_2x}
\]

\[
y'' = 4ae^{r_1x} + 4be^{r_2x}
\]

left side of the equation = \( y'' - 4y \)

\[
= 4ae^{r_1x} + 4be^{r_2x} - 4(ae^{r_1x} + be^{r_2x})
\]

\[
= 4ae^{r_2x} + 4be^{r_2x} - 4ae^{r_1x} + 4be^{r_1x}
\]

\[
= 0
\]
4. (10 points) Use Euler’s method with step size $h = 0.2$ to solve $\frac{dy}{dx} = 2xy - 1$ with initial value $y(1) = 1$. Find $y(1.6)$.

$x_0 = 1 \quad y_0 = 1$

$x_1 = x_0 + h = 1.2 \quad y_1 = y_0 + h \left(2x_0 y_0 - 1\right) = 1 + 0.2 \left(2\cdot1\cdot1 - 1\right) = 1.2$

$x_2 = x_1 + h = 1.4 \quad y_2 = y_1 + h \left(2x_1 y_1 - 1\right) = 1.2 + 0.2 \left(2\cdot1.2\cdot1.2 - 1\right) = 1.576$

$x_3 = x_2 + h = 1.6 \quad y_3 = y_2 + h \left(2x_2 y_2 - 1\right) = 1.576 + 0.2 \left(2\cdot1.4\cdot1.576 - 1\right) = 2.25856$

So, $y(1.6) \approx y_3 = 2.25856$
5. (10 points) Solve the differential equation \( \frac{dy}{dx} = 4y \cos x \) with the initial condition \( y(0) = 3 \).

\[
\frac{dy}{y} = 4 \cos x \, dx
\]

\[
\int \frac{1}{y} \, dy = 4 \int \cos x \, dx
\]

\[
\ln |y| = 4 \sin x + C
\]

\[
|y| = e^{4 \sin x + C} = e^{4 \sin x} \cdot e^C
\]

\[
y = \pm e^{4 \sin x} \cdot e^C
\]

Denote \( a = \pm e^C \),

\[
y = a \cdot e^{4 \sin x}
\]

By initial condition \( y(0) = 3 \),

\[
3 = a \cdot e^{4 \sin 0} = a
\]

So, the solution is \( y = 3 e^{4 \sin x} \).

6. (2 bonus points) Solve the differential equation \( (\sec^2 y)x^{-2} y' = \sin(2x^3) \)

\[
\sec^2 y \, dy = x^2 \sin 2x^3 \, dx
\]

\[
\int \sec^2 y \, dy = \int x^2 \sin 2x^3 \, dx
\]

\[
\tan y = -\frac{1}{6} \cos 2x^3 + C
\]