11.1 Sequences and Limits

**Definition**

A **sequence** is a list of order numbers

\[ \{a_1, a_2, a_3, a_4, \ldots, a_n, \ldots \} , \]

which is denoted by

\[ \{a_n\}_{n=1}^{\infty} \text{, or } \{a_n\} \text{ for short}. \]

**Example**

(1). \( \{a_n\}_{n=1}^{\infty} \) is \( \{2n - 1\} \)

(2). \( \{b_n\}_{n=1}^{\infty} \) is \( \left\{ \frac{(-1)^n n}{e^n} \right\} \)

(3). \( \{c_n\}_{n=1}^{\infty} \) is \( \left\{ \sqrt{3n} \right\} \)
Definition (Limits)

If a sequence \( \{a_n\} \) has the limit \( L \) as \( n \to \infty \), we write

\[
\lim_{n \to \infty} a_n = L,
\]

and say \( \{a_n\} \) converges to \( L \).

If \( \{a_n\} \) has no limit, we say \( \{a_n\} \) diverges.

Theorem

If \( \lim_{x \to \infty} f(x) = L \) and \( f(n) = a_n \) when \( n \) is an integer, then,

\[
\lim_{n \to \infty} f(n) = L.
\]
Definition (Limit \( L \) (Precise Definition. ) )

A sequence \( \{ a_n \} \) has \textbf{limit} \( L \) as \( n \to \infty \), if for every \( \epsilon > 0 \) there is a corresponding integer \( N \) such that

\[
\text{if } \quad n > N, \quad \text{then } |a_n - L| < \epsilon.
\]

Definition (Limit \( \infty \) (Precise Definition. ) )

A sequence \( \{ a_n \} \) has \textbf{limit} \( \infty \) as \( n \to \infty \), denoted as \( \lim_{n \to \infty} a_n = \infty \), if for every positive number \( M \) there is an integer \( N \) such that

\[
\text{if } \quad n > N, \quad \text{then } a_n > M.
\]
The Limit Laws for functions hold for the limits of sequences

**Theorem (Limit Laws)**

If \( \{a_n\} \) and \( \{b_n\} \) are convergent and \( k \) is a constant number, then

\[
\lim_{n \to \infty} (a_n + b_n) = \lim_{n \to \infty} a_n + \lim_{n \to \infty} b_n
\]

\[
\lim_{n \to \infty} ka_n = k \lim_{n \to \infty} a_n
\]

\[
\lim_{n \to \infty} (a_nb_n) = \lim_{n \to \infty} a_n \cdot \lim_{n \to \infty} b_n
\]

\[
\lim_{n \to \infty} \frac{a_n}{b_n} = \frac{\lim_{n \to \infty} a_n}{\lim_{n \to \infty} b_n}
\]

\[
\lim_{n \to \infty} (a_n)^p = (\lim_{n \to \infty} a_n)^p \text{ if } p > 0 \text{ and } a_n > 0.
\]
The Squeeze Theorem: If \( a_n \leq b_n \leq c_n \) for \( n \geq N \), and \( \lim_{n \to \infty} a_n = \lim_{n \to \infty} c_n = L \), then
\[
\lim_{n \to \infty} b_n = L
\]

If \( \lim_{n \to \infty} |a_n| = 0 \), then \( \lim_{n \to \infty} a_n = 0 \)

**Theorem.** If \( \lim_{n \to \infty} a_n = L \) and the function \( f(x) \) is continuous at \( x = L \) then
\[
\lim_{n \to \infty} f(a_n) = f(L).
\]
Definition (Monotonic)

A sequence \( \{a_n\} \) is called **increasing** if \( a_n < a_{n+1} \) for all \( n \geq 1 \), that is

\[
a_1 < a_2 < a_3 < \cdots.
\]

A sequence \( \{a_n\} \) is called **decreasing** if \( a_n > a_{n+1} \) for all \( n \geq 1 \), that is

\[
a_1 > a_2 > a_3 > \cdots.
\]

A sequence \( \{a_n\} \) is called **monotonic** if it is either increasing or decreasing.
Definition (Bounded)

A sequence \( \{a_n\} \) is called **bounded above** if there is a number \( M \) such that
\[
a_n \leq M \text{ for all } n \geq 1
\]
A sequence \( \{a_n\} \) is called **bounded below** if there is a number \( m \) such that
\[
m \leq a_n \text{ for all } n \geq 1.
\]
A sequence \( \{a_n\} \) is called **bounded sequence** if it is bounded above and below.

Monotonic Sequence Theorem

Every *bounded, monotonic* sequence is convergent.