Find the volume of a solid $S$ between $a \leq x \leq b$.

1. The volume of the cylinder is defined as

$$V = Ah$$

where, the area of the base is $A$ and the height of the cylinder is $h$.

The **cross-section** of $S$ is the intersection of a plane $P_x$ perpendicular to $x$-axis with $S$.

Let $A(x)$ be the area of the cross-section.

(or $P_y$ perpendicular to y-axis)
We divide \([a, b]\) by \(n\) parts. Then we also divide the solid \(S\) by \(n\) parts \(S_i\). Each part \(S_i\) can be approximated by a cylinder, with volume

\[ V(S_i) \approx A(x_i^*) \Delta x. \]

The volume of \(S\) can be approximated by the sum

\[ V(S) \approx \sum_{i=1}^{n} A(x_i^*) \Delta x. \]

If \(A(x)\) is a continuous function, the **volume** of \(S\) is defined by the limit of Riemann sums

\[ V = \lim_{n \to \infty} \sum_{i=1}^{n} A(x_i^*) \Delta x = \int_{a}^{b} A(x) dx. \]

**Example 1.** Let \(S\) be a cone of radius \(r\) and height \(h\). Show that the volume of \(S\) is \(V = \frac{1}{3} \pi r^2 h\).

\[ 0 \leq x \leq h \]

\[ \frac{y}{r} = \frac{x}{h} \quad y = \frac{rx}{h} \]

\[ A(x) = \pi y^2 = \frac{\pi r^2 x^2}{h^2} \]

\[ \text{Volume} = \int_{0}^{h} \frac{\pi r^2 x^2}{h^2} \, dx \]

\[ = \left. \frac{\pi r^2}{h^2} \cdot \frac{x^3}{3} \right|_{0}^{h} \]

\[ = \frac{1}{3} \pi r^2 h. \]
Example 2. Let $S$ be a solid obtained by rotating about the $x$-axis the region under the curve $y = \sqrt{x}$ from 0 to 2.

$$A(x) = \pi y^2 = \pi (\sqrt{x})^2 = \pi x$$

$$\text{Volume} = \int_{0}^{2} A(x) \, dx$$

$$= \int_{0}^{2} \pi x \, dx$$

$$= \pi \left. \frac{x^2}{2} \right|_{0}^{2}$$

$$= 2\pi$$

Example 3. Let $S$ be a solid obtained by rotating about the $y$-axis the region bounded by $y = x^4$, $y = 4$. $0 \leq y \leq 4$

$$A(y) = \pi x^2$$

$$= \pi \left( \frac{y}{4} \right)^2$$

$$= \pi \frac{y^2}{16}$$

$$\text{Volume} = \int_{0}^{4} A(y) \, dy$$

$$= \int_{0}^{4} \pi \frac{y^2}{4} \, dy$$

$$= \frac{3}{3} \pi \left. \frac{y^3}{2} \right|_{0}^{4}$$

$$= \frac{16}{3} \pi$$
Example 4. The region $R$ is in the first quadrant enclosed by the curves $y = x$ and $y = x^3$. Find the volume of the solid obtained by rotating $R$ about the $x$-axis.

\[
A(x) = \pi x^2 - \pi (x^3)^2
\]
\[
= \pi (x^2 - x^6)
\]

Volume = \[
\int_0^1 A(x) \, dx
\]
\[
= \int_0^1 \pi (x^2 - x^6) \, dx
\]
\[
= \left[ \pi \left( \frac{x^3}{3} - \frac{x^7}{7} \right) \right]_0^1
\]
\[
= \pi \left( \frac{1}{3} - \frac{1}{7} \right) = \frac{4}{21} \pi
\]

Example 5. Find the volume of the solid obtained by rotating the region in Example 4 about the line $y = 2$.

\[
A(x) = \pi (2-x^3)^2 - \pi (2-x)^2
\]
\[
= \pi \left( 4x^3 - 8x^2 + 4x \right)
\]

Volume = \[
\int_0^1 A(x) \, dx
\]
\[
= \int_0^1 \pi \left( 4x^3 - 8x^2 + 4x \right) \, dx
\]
\[
= \left[ \pi \left( \frac{x^4}{4} - \frac{8x^3}{3} + \frac{4x^2}{2} \right) \right]_0^1
\]
\[
= \pi \left( \frac{1}{4} - \frac{8}{3} + 2 \right) = \frac{11}{24} \pi
\]
Example 6. Find the volume of the solid obtained by rotating the region in Example 4 about $y$-axis.

\[
A(y) = \pi \left( y^{3/2} \right)^2 - \pi(y^2)
\]

\[
\text{Volume} = \int_0^1 A(y) \, dy
\]

\[
= \int_0^1 \pi \left( y^{3/2} - y^2 \right) \, dy
\]

\[
= \pi \left( \frac{2}{3} y^{5/2} - \frac{2}{3} y^3 \right) \bigg|_0^1
\]

\[
= \frac{4}{15} \pi
\]

Example 7. Find the volume of the solid obtained by rotating the region in Example 4 about the line $x = 1$.

\[
A(y) = \pi \left( 1-y \right)^2 - \pi \left( 1 - y^{3/2} \right)^2
\]

\[
\text{Volume} = \int_0^1 A(y) \, dy
\]

\[
= \int_0^1 \pi \left( y^2 - 2y - y^3 + 2 y^{3/2} \right) \, dy
\]

\[
= \pi \left( \frac{y^3}{3} - y^2 - \frac{3}{5} y^{5/2} + \frac{3}{2} y^{3/2} \right) \bigg|_0^1
\]

\[
= \pi \left( \frac{1}{3} - 1 - \frac{3}{5} + \frac{3}{2} \right)
\]

\[
= \frac{7}{30} \pi
\]