Example 1. The region $R$ is in the first quadrant enclosed by the curves $y = 3x^2 - x^3$ and $y = 0$. Find the volume of the solid obtained by rotating $R$ about the $y$-axis.

$$A = \pi (\text{outer radius})^2 - \pi (\text{inner radius})^2$$

we need to solve $x = (?)$, but we can't.

Method of cylindrical shells:

$$V = V_{\text{out}} - V_{\text{in}}$$

$$= \pi r_2^2 h - \pi r_1^2 h$$

$$= \pi (r_2^2 - r_1^2) h$$

$$= \pi (r_2 + r_1)(r_2 - r_1) h$$

$$= 2\pi \left( \frac{r_2 + r_1}{2} \right) h \frac{(r_2 - r_1)}{r}$$

The volume of a cylindrical shell is

$$V = 2\pi rh \Delta r$$
The volume of the solid is given by the sum of the volumes of the shells

\[ V \approx \sum_{i=1}^{n} 2\pi x_i^* f(x_i^*) \Delta x. \]

The volume of the solid \( S \) obtained by rotating about the \( y \)-axis the region \( R \) under the curve \( y = f(x) \) from \( a \) to \( b \), is

\[ V = \lim_{n \to \infty} \sum_{i=1}^{n} 2\pi x_i^* f(x_i^*) \Delta x. \]

Hence, using the definition of definite integral

\[ V = \int_{a}^{b} 2\pi x f(x) dx \]

**Example 1.**

\[ 3x^2 - x^3 = 0 \]

\[ x^2(3-x) = 0 \]

\[ x = 0 \text{ or } 3 \]

\[ V = \int_{0}^{3} 2\pi x \left( 3x^2 - x^3 \right) dx \]

\[ = \int_{0}^{3} 6\pi x^3 - 2\pi x^4 dx \]

\[ = \left. \frac{6\pi x^4}{4} - \frac{2\pi x^5}{5} \right|_{0}^{3} \]

\[ = \frac{243 \pi}{10} \]
Example 2. The region $R$ is in the first quadrant enclosed by the curves $y = x$ and $y = x^3$. Use cylindrical shells, find the volume of the solid obtained by rotating $R$ about the $y$-axis.

\[ x = x^3 \]
\[ x(1-x^2) = 0 \]
\[ x = 0 \text{ or } x = 1 \]

\[ V = \int_0^1 2\pi x (x - x^3) \, dx \]
\[ = \int_0^1 2\pi x^2 - 2\pi x^4 \, dx \]
\[ = 2\pi \left[ \frac{x^3}{3} - \frac{x^5}{5} \right]_0^1 = 2\pi \left( \frac{1}{3} - \frac{1}{5} \right) = 2\pi \left( \frac{2}{15} \right) \]

Example 3. The region $R$ is enclosed by the curves $x = 2y^2 - y^3$ and $x = 0$. Use cylindrical shells, find the volume of the solid obtained by rotating $R$ about the $x$-axis.

\[ 2y^2 - y^3 = 0 \]
\[ y^2(2-y) = 0 \quad y = 0 \text{ or } y = 2 \]

\[ V = \int_0^2 2\pi y (2y^2 - y^3) \, dy \]
\[ = \int_0^2 2\pi y^2 - 2\pi y^4 \, dy \]
\[ = \pi y^4 - \frac{2\pi y^5}{5} \bigg|_0^2 = \frac{16\pi}{5} \]
Example 4. Find the volume of the solid obtained by rotating the region bounded by \( y = 2x - x^2 \) and \( y = 0 \) about the line \( x = 3 \).

\[ 2x - x^2 = 0 \]
\[ x(2-x) = 0 \quad x = 0 \quad \text{or} \quad x = 2 \]

\[ V = \int_0^2 2\pi (3-x)(2x-x^2) \, dx \]
\[ = 2\pi \int_0^2 6x - 5x^2 + x^3 \, dx \]

Example 5. The region \( R \) is in the first quadrant enclosed by the curves \( xy = 2 \), \( x = 0 \), \( y = 3 \), \( y = 4 \). Use cylindrical shells method to find the volume of the solid obtained by rotating \( R \) about the \( x \)-axis.

\[ V = \int_3^4 2\pi y \left( \frac{2}{y} \right) \, dy \]
\[ = \int_3^4 4\pi \, dy \]
\[ = 4\pi y \Big|_3^4 \]
\[ = 4\pi \]