We know how to calculate the length of a line segment between \(A(x_1, y_1)\) and \(B(x_2, y_2)\) by Pythagorean Theorem:

\[
|AB| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}
\]

**Question:** How to calculate the length of a curve \(C\) defined by equation \(y = f(x)\), with \(a \leq x \leq b\)?

The length \(L\) of \(C\) is approximately the sum of the lengths of line segments

\[
L \approx \sum_{i=1}^{n} |P_{i-1}P_i|.
\]

Here,

\[
|P_{i-1}P_i| = \sqrt{(x_i - x_{i-1})^2 + (y_i - y_{i-1})^2}
\]

\[
= \sqrt{\Delta x_i^2 + \Delta y_i^2}
\]

\[
= \sqrt{1 + \left(\frac{\Delta y_i}{\Delta x_i}\right)^2} \Delta x_i
\]

\[
\approx \sqrt{1 + [f'(x_i)]^2} \Delta x_i
\]
The more line segments, the better approximating. To make it precise, we take the limit:

\[
L = \lim_{n \to \infty} \sum_{i=1}^{n} |P_{i-1}P_i|
= \lim_{n \to \infty} \sum_{i=1}^{n} \sqrt{1 + [f'(x_i)]^2} \Delta x_i
= \int_{a}^{b} \sqrt{1 + [f'(x)]^2} \, dx
\]

**The Arc Length Formula**

If \( f'(x) \) is continuous on \([a, b]\), then the length of the curve \( y = f(x) \) on \([a, b]\) is

\[
L = \int_{a}^{b} \sqrt{1 + [f'(x)]^2} \, dx
\]

**Example 1.** Use the arc length formula to find the length of the curve \( y = 3x - 5 \) when \( 1 \leq x \leq 3 \). Check your answer by Pythagorean Theorem.

1. \( y' = 3 \)

\[
\begin{align*}
\theta & \quad L = \int_{1}^{3} \sqrt{1 + 3^2} \, dx = \int_{1}^{3} \sqrt{10} \, dx = \sqrt{10}(3-1) = 2\sqrt{10} \\
\end{align*}
\]

Arc length formula.

2. Pythagorean Theorem:

\[
\begin{align*}
\begin{array}{cccc}
\chi_{1} & \chi_{2} & \chi_{3} & y_{1} = -2 \\
& & y_{2} = 4 \\
\end{array}
\end{align*}
\]

\[
L = \sqrt{(\chi_{1}-\chi_{2})^2 + (y_2 - y_1)^2} = \sqrt{(3-1)^2 + (4-(-2))^2} = \sqrt{40} = 2\sqrt{10}
\]
Example 2. Use the arc length formula to find the length of the curve \( y = 2x^{3/2} - 3 \), for \( 1 \leq x \leq 4 \).

1. \( y' = 2 \left( \frac{3}{2} \right) x^{1/2} = 3x^{1/2} \)

2. By arc length formula

\[
L = \int_1^4 \sqrt{1 + (3x^{1/2})^2} \, dx = \int_1^4 \sqrt{1 + 9x} \, dx
\]

Let \( u = 1 + 9x \), \( u(1) = 10 \)
\( du = 9 \, dx \), \( u(4) = 37 \)
\( dx = \frac{1}{9} \, du \)

\[
= \frac{1}{9} \cdot \frac{u^{3/2}}{3/2} \Bigg|_{10}^{37} = \frac{27}{27} \left( 37^{3/2} - 10^{3/2} \right)
\]

Example 3. Use the arc length formula to find the length of the curve \( y = 2(x - 3)^{3/2} \), for \( 3 \leq x \leq 4 \).

1. \( y' = 3(x-3)^{1/2} \)

2. By arc length formula

\[
L = \int_3^4 \sqrt{1 + (y')^2} \, dx = \int_3^4 \sqrt{1 + 9(x-3)} \, dx
\]

Let \( u = 9x - 26 \)
\( du = 9 \, dx \)
\( dx = \frac{1}{9} \, du \)

\[
= \frac{1}{9} \cdot \frac{u^{3/2}}{3/2} \Bigg|_3^{10} = \frac{27}{27} \left( 10^{3/2} - 1 \right)
\]

\[
= \frac{27}{27} \left( 10^{3/2} - 1 \right)
\]
Similarly,

If \( x = g'(y) \) is continuous on \( c \leq y \leq d \), then the length of the curve \( x = g(y) \) on \( c \leq y \leq d \) is

\[
L = \int_c^d \sqrt{1 + [g'(y)]^2} \, dy
\]

**Example 4.** Use the arc length formula to find the length of the curve \( y^3 = x^2 \), for \( 1 \leq y \leq 4 \).

\[\begin{align*}
0 & \quad x = y^{\frac{3}{2}} \\
2 & \quad x' = \frac{dx}{dy} = \frac{3}{2} y^{\frac{1}{2}} \\
3 & \quad \text{By arc length formula}.
\end{align*}\]

\[
L = \int_1^4 \sqrt{1 + (x')^2} \, dy = \int_1^4 \sqrt{1 + \frac{9}{4} y} \, dy
\]

Let \( u = 1 + \frac{9}{4} y \)

\[
\begin{align*}
\frac{du}{dy} & = \frac{9}{4} \\
\frac{dy}{du} & = \frac{4}{9} \\
u(1) & = \frac{13}{4} \\
u(4) & = 10
\end{align*}
\]

\[
L = \int_{\frac{13}{4}}^{10} u^{\frac{1}{2}} \cdot \frac{4}{9} \, du = \left[ \frac{8}{27} u^{\frac{3}{2}} \right]_{\frac{13}{4}}^{10} = \frac{8}{27} \left( 10^{\frac{3}{2}} - \left(\frac{13}{4}\right)^{\frac{3}{2}} \right)
\]

Remark: Use \( y = x^{2/3} \) will be hard to calculate.
The Arc Length Function:
It is useful to define a function \( s(x) \) measuring the arc length of a curve from starting point \( t = a \) to any other point \( t = x \) on the curve \( C \).

If \( y = f'(t) \) is continuous on \([a, b]\), then the Arc Length Function is defined as

\[
s(x) = \int_a^x \sqrt{1 + [f'(t)]^2} \, dt
\]

Here, \( x \) is the variable for the arc length function.

The Fundamental Theorem of Calculus gives

\[
\frac{ds}{dx} = \sqrt{1 + [f'(x)]^2} = \sqrt{1 + \left( \frac{dy}{dx} \right)^2}
\]

Take squares both sides, it can also be written as

\[
(ds)^2 = (dx)^2 + (dy)^2
\]

Example 5. Find the arc length function for the curve \( y = \frac{1}{4}x^2 - \frac{1}{2} \ln x \) taking \((1, 1/4)\) as the starting point.

\[
y' = \frac{1}{2}x - \frac{1}{2x}
\]

\[
1 + (y')^2 = 1 + \left( \frac{x}{2} - \frac{1}{2x} \right)^2 = \left( \frac{x}{2} + \frac{1}{2x} \right)^2
\]

\[
S(x) = \int_1^x \sqrt{1 + (y')^2} \, dt
\]

\[
= \int_1^x \sqrt{1 + \left( \frac{x}{2} + \frac{1}{2x} \right)^2} \, dt
\]

\[
= \left. \frac{t^2}{4} + \frac{1}{2} \ln t \right|_1^x
\]

\[
= \frac{x^2}{4} + \frac{1}{2} \ln x - \frac{1}{4}
\]