It is very hard or even impossible to find the \textit{precise solutions} for most differential equations from our real world problem. So, it is useful to get an estimation for the solutions using \underline{graphical approach or a numerical approach.}

1. Graphical approach (Slope fields, or direction fields)

If \( y = f(x) \) is a solution for the differential equation \( \frac{dy}{dx} = F(x, y) \), then the \textit{slope of curve} \( y = f(x) \) at \((x, y)\) is \( F(x, y) \). At each point \((x, y)\) draw a line segment with slope \( F(x, y) \). The solution is the curve tangent at this point.

\textbf{Example 1.} \( \frac{dy}{dx} = x - y \) (Use slope fields).

Three solutions with initial values \( y(0) = -3, \quad y(0) = -1, \quad y(0) = 1 \) are drawn in the graph.
2. Numerical approach (Euler’s Method)

Find approximating solution for the differential equation \( \frac{dy}{dx} = F(x, y) \) with initial value \( y(x_0) = y_0 \) using Euler’s Method with step size \( h \):

1. Set step size \( h \); (the smaller \( h \), the better estimation.)
2. Start with point \((x_0, y_0)\);
3. Define a sequence \( x_n = x_{n-1} + h \);
4. Then \( y_n \) is computed by the sequence

   \[
   y_n = y_{n-1} + hF(x_{n-1}, y_{n-1})
   \]

Example 2. Use Euler’s method with step size \( h = 0.5 \) to solve \( \frac{dy}{dx} = x - y \) with initial value \( y(0) = 1 \).

\[
\begin{align*}
    x_0 &= 0 \\
    y_0 &= 1 \\
    x_1 &= x_0 + h = 0.5 \\
    y_1 &= y_0 + h(x_0 - y_0) = 1 + 0.5(0 - 1) = 0.5 \\
    x_2 &= x_1 + h = 1 \\
    y_2 &= y_1 + h(x_1 - y_1) = 0.5 + 0.5(0.5 - 0.5) = 0.5 \\
    x_3 &= x_2 + h = 1.5 \\
    y_3 &= y_2 + h(x_2 - y_2) = 0.5 + 0.5(1 - 0.5) = 0.75 \\
    x_4 &= x_3 + h = 2 \\
    y_4 &= y_3 + h(x_3 - y_3) = 0.75 + 0.5(1.5 - 0.75) = 1.125 \\
    x_5 &= x_4 + h = 2.5 \\
    y_5 &= y_4 + h(x_4 - y_4) = 1.125 + 0.5(2 - 1.125) = 1.5625
\end{align*}
\]

So \( y(2.5) \approx y_5 = 1.5625 \)
Example 3. Use Euler’s method with step size \( h = 0.1 \) to solve \( \frac{dy}{dx} = xy \) with initial value \( y(1) = 1 \). Find \( y(1.4) \) and \( y(1.5) \).

\[
x_0 = 1 \\
y_0 = 1 \\
x_1 = x_0 + h = 1.1 \\
y_1 = y_0 + h(x_0y_0) = 1 + 0.1(1) = 1.1 \\
x_2 = x_1 + h = 1.2 \\
y_2 = y_1 + h(x_1y_1) = 1.1 + 0.1(1.1)(1.1) = 1.221 \\
x_3 = x_2 + h = 1.3 \\
y_3 = y_2 + h(x_2y_2) = 1.221 + 0.1(1.2)(1.221) = 1.36752 \\
x_4 = x_3 + h = 1.4 \\
y_4 = y_3 + h(x_3y_3) = 1.36752 + 0.1(1.3)(1.36752) = 1.5452976 \\
x_5 = x_4 + h = 1.5 \\
y_5 = y_4 + h(x_4y_4) = 1.5452976 + 0.1(1.4)(1.5452976) \\
= 1.761639264.
\]

So, \( y(1.4) \approx y_4 = 1.5452976 \)

\( y(1.5) \approx y_5 = 1.761639264 \)

In fact, from Example 2 in §9.3, we know the solutions are

\[
y = e^{\frac{1}{2}x^2} = e^{\frac{1}{2}(x^2-1)}
\]

So, \( y(1.5) = e^{0.625} \approx 1.868 \)