1. (5 points) Find all second order partial derivatives of the function \( f(x,y) = x^2e^y - x\cos 2y \).

\[
\begin{align*}
    f_x &= 3x^2e^y - \cos 2y \\
    f_y &= x^2e^y + 2x\sin 2y \\
    f_{xx} &= 6xe^y \\
    f_{yy} &= x^2e^y + 4x\cos 2y \\
    f_{xy} &= f_{yx} = 3x^2e^y + 2\sin 2y
\end{align*}
\]

2. (6 points) Let \( z = \ln(2x + 3y) \), where \( x = s\cos t \) and \( y = st^2 \). Use the chain rule to find \( \frac{\partial z}{\partial t} \).

(Write your result as a function with variables \( s \) and \( t \).)

\[
\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial t}
\]

\[
\begin{align*}
    &= \frac{2}{2x + 3y} \cdot (-s\sin t) + \frac{3}{2x + 3y} (2st) \\
    &= \frac{-2s\sin t}{2s\cos t + 3st^2} + \frac{6st}{2s\cos t + 3st^2}
\end{align*}
\]

3. (3 points) Find the limits of the following two functions. If it does not exist, write No Limit.

\[
\begin{align*}
    \lim_{(x,y) \to (0,0)} \frac{xy}{x^2 + y^2} &= \text{No Limit} \\
    \lim_{(x,y) \to (0,1)} \frac{xy}{x^2 + y^2} &= 0 \\
    \lim_{(x,y) \to (0,0)} \frac{x}{y} &= \text{No Limit}
\end{align*}
\]
4. Let \( f(x,y) = x \ln y + 3y \).

(1) (4 points). Find the gradient of \( f(x,y) \).

\[
\nabla f(x,y) = \langle f_x, f_y \rangle = \langle \ln y, \frac{x}{y} + 3 \rangle
\]

(2) (1 point). Find the gradient of \( f(x,y) \) at \((2,1)\).

\[
\nabla f(2,1) = \langle \ln 1, \frac{2}{1} + 3 \rangle = \langle 0, 5 \rangle
\]

(3) (3 points). Find the directional derivative of \( f(x,y) \) at \((2,1)\) in the direction of the vector \( \vec{v} = (3,4) \).

\[
\vec{u}^2 = \frac{\vec{v}^2}{\|\vec{v}\|} = \frac{1}{\sqrt{25}} \langle 3, 4 \rangle = \langle \frac{3}{5}, \frac{4}{5} \rangle
\]

\[
\nabla f(2,1) \cdot \vec{u} = \langle 0, 5 \rangle \langle \frac{3}{5}, \frac{4}{5} \rangle = 4
\]

(4) (2 points). In which direction does the function \( f(x,y) \) at \((2,1)\) have the maximum rate of change.

\[
\nabla f(2,1) = \langle 0, 5 \rangle
\]

(5) (2 points). Find the maximum rate of change of \( f(x,y) \) at \((2,1)\).

\[
| \nabla f(2,1) | = \sqrt{0^2 + 5^2} = 5
\]
5. Consider the surface $z = f(x, y) = y^2 e^x$ at the point $P(0, 2, 4)$.

(1) (6 points). Find an equation of the **tangent plane** to the surface $z = y^2 e^x$ at the point $P(0, 2, 4)$.

\[
\begin{align*}
    f_x &= y^2 e^x \\
    f_x(0, 2) &= 4 \\
    f_y &= 2y e^x \\
    f_y(0, 2) &= 4
\end{align*}
\]

The tangent plane at $(0, 2, 4)$ is

\[
z - 4 = 4x + 4(y - 2)
\]

or \[z = 4x + 4y - 4\]

(2) (2 points). Find the **linearization** $L(x, y)$ of the function $f(x, y) = y^2 e^x$ at the point $(0, 2)$.

\[
L(x, y) = 4 + 4x + 4(y - 2)
\]

\[
= 4x + 4y - 4
\]

(3) (4 points). Find an equation of the **normal line** to the surface $z = y^2 e^x$ through the point $P(0, 2, 4)$. (Hint: the normal line is the line which is perpendicular to the tangent plane at $P$.)

**Normal vector** $\mathbf{n} = \langle 4, 4, -1 \rangle$

**Normal line at $P(0, 2, 4)$** is

\[
\frac{x}{4} = \frac{y - 2}{4} = \frac{z - 4}{-1}
\]
6. Let \( f(x, y) = x^2 + y^3 - 4x - 3y^2 + 5 \).

(1)(4 points). Find all critical points of \( f(x, y) \).

\[
\begin{align*}
    f_x &= 2x - 4 = 0 \quad \Rightarrow x = 2 \\
    f_y &= 3y^2 - 6y = 0 \quad \Rightarrow y^2 = 2y \quad \Rightarrow y = 0 \text{ or } y = 2
\end{align*}
\]

Critical points (2, 0) and (2, 2)

(2)(8 points). Use the second derivative test to determine the local maximum, the local minimum, and saddle point of the function \( f(x, y) \).

\[
\begin{align*}
    f_{xx} &= 2 \\
    f_{xy} &= 0 \\
    f_{yy} &= 6y - 6
\end{align*}
\]

\[
\begin{bmatrix}
    f_{xx} & f_{xy} \\
    f_{yx} & f_{yy}
\end{bmatrix} = \begin{bmatrix}
    2 & 0 \\
    0 & 6y - 6
\end{bmatrix} = 12(y - 1)
\]

For (2, 0), \( D(2, 0) = -12 < 0 \) so (2, 0) is a saddle point.

For (2, 2), \( D(2, 2) = 12 > 0 \) and \( f_{xx} = 2 > 0 \) so (2, 2) is a local minimum.

**Bonus question** (2 points): Find the absolute maximum and minimum values of the function \( f(x, y) = x^2 + y^2 - 2x - 2y - 1 \) on the rectangle \( D = \{(x, y) \mid 0 \leq x \leq 3, 0 \leq y \leq 2\} \).

Answer: Absolute maximum: \( 2 \) \quad Absolute minimum: \( -3 \)