§12.2 Vectors

A vector is a quantity that has magnitude and direction. A vector is often represented by a directed line segment, denoted by $\mathbf{v}$ or $\mathbf{u}$. A particle moves along a line segment from point $A$ to point $B$.

* The vector $\mathbf{AB}$ has the same length and the same direction as $\mathbf{CD}$ even though it is in a different position. We say that $\mathbf{AB}$ and $\mathbf{CD}$ are equivalent (or equal) and we write $\mathbf{AB} = \mathbf{CD}$.

* Zero vector has no direction, denoted by $\mathbf{0}$.

- Operations of vectors

1. Sum $\mathbf{AB} + \mathbf{BC} = \mathbf{AC}$.

The triangle law

The parallelogram law

$$\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$$
2. **Scalar product**

If \( c \in \mathbb{R} \) is a scalar and \( \vec{v} \) is a vector, then the **scalar multiple** \( c\vec{v} \) is the vector whose length \( |c| \) is times the length of \( \vec{v} \) and whose direction is the same as \( \vec{v} \) if \( c > 0 \) and is opposite to \( \vec{v} \) if \( c < 0 \).

If \( c = 0 \) or \( \vec{v} = \vec{0} \), then \( c\vec{v} = \vec{0} \).

3. **Difference**

Difference of two vectors can be defined using sum and scalar product: \( \vec{u} - \vec{v} = \vec{u} + (-1)\vec{v} \)
Components (vectors in coordinate system)

- A vector \( \vec{v} \) starting from origin to a point \( P((a, b) \text{ or } (a, b, c)) \), depending on \( \mathbb{R}^2 \text{ or } \mathbb{R}^3 \) is called the **position vector** of \( P \). The coordinates are called the **components** of \( \vec{v} \). We denote \( \vec{v} = \langle a, b, c \rangle \).
- Given the points \( A(x_1, x_2, x_3) \) and \( B(x_2, y_2, z_2) \), the vector \( \vec{v} \) with representation \( \overrightarrow{AB} \) is
  \[
  \vec{v} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle.
  \]

**Example:** Find the vector represented by the directed line segment with initial point \( A(1, -2, 3) \) and terminal point \( B(2, 1, 5) \).

*The vector corresponding to \( \overrightarrow{AB} \) is*

\[
\vec{v} = \langle 2-1, 1-(-2), 5-3 \rangle = \langle 1, 3, 2 \rangle
\]

The magnitude or length of the vector \( v = \langle a, b, c \rangle \) in \( \mathbb{R}^3 \) is

\[
|\vec{v}| = \sqrt{a^2 + b^2 + c^2}.
\]

The length of \( v = \langle a, b \rangle \) in \( \mathbb{R}^2 \) is \( |\vec{v}| = \sqrt{a^2 + b^2} \).

**Q:** How to add and subtract algebraic vectors? **A:** Component-wise.

If \( \vec{v} = \langle a_1, a_2 \rangle \) and \( \vec{w} = \langle b_1, b_2 \rangle \), then

\[
\vec{v} + \vec{w} = \langle a_1 + b_1, a_2 + b_2 \rangle; \quad \vec{v} - \vec{w} = \langle a_1 - b_1, a_2 - b_2 \rangle; \quad \text{and} \quad c\vec{v} = \langle ca_1, ca_2 \rangle.
\]
Example: If \( \vec{a} = \langle 2, 3, 0 \rangle \) and \( \vec{b} = \langle -1, 2, 4 \rangle \), then
\[
|\vec{a}| = \sqrt{2^2 + 3^2} = \sqrt{13}
\]
\[
\vec{a} + \vec{b} = \langle 1, 5, 4 \rangle
\]
\[
3\vec{b} = \langle -3, 6, 12 \rangle
\]
\[
\vec{a} - \vec{b} = \langle 3, 1, -4 \rangle
\]
\[
2\vec{a} + 3\vec{b} = \langle 4, 6, 0 \rangle + \langle -3, 6, 12 \rangle = \langle 1, 12, 12 \rangle
\]

**Theorem (Algebraic Properties)** For \( \vec{u} \) and \( \vec{w} \) vectors in \( \mathbb{R}^n \), and \( c, d \) scalars, the following algebraic properties hold.

1. \( \vec{u} + \vec{w} = \vec{w} + \vec{u} \) \textit{Commutative}
2. \((\vec{u} + \vec{v}) + \vec{w} = \vec{v} + (\vec{u} + \vec{w}) \) \textit{Associative}
3. \( \vec{u} + \vec{0} = \vec{u} \)
4. \( \vec{u} + (-\vec{u}) = \vec{0} \)
5. \( c(\vec{u} + \vec{w}) = c\vec{u} + c\vec{w} \) \textit{Distributive}
6. \( (c + d)\vec{u} = c\vec{u} + d\vec{u} \)
7. \( c(d\vec{u}) = (cd)\vec{u} \)
8. \( 1\vec{u} = \vec{u} \)

**Standard basis vectors:** \( \vec{i} = \langle 1, 0, 0 \rangle, \vec{j} = \langle 0, 1, 0 \rangle, \vec{k} = \langle 0, 0, 1 \rangle \).
We can express any vector \( \vec{v} = \langle a, b, c \rangle \) in terms of \( \vec{i}, \vec{j}, \) and \( \vec{k}, \) as

\[
\vec{v} = a\vec{i} + b\vec{j} + c\vec{k}.
\]

\[
\vec{v} = \langle a, b, c \rangle = \langle a, 0, 0 \rangle + \langle 0, b, 0 \rangle + \langle 0, 0, c \rangle
\]

\[
= a\langle 1, 0, 0 \rangle + b\langle 0, 1, 0 \rangle + c\langle 0, 0, 1 \rangle
\]

\[
= a\vec{i} + b\vec{j} + c\vec{k}
\]

**Example:** If \( \vec{a} = 2\vec{i} + \vec{j} - 4\vec{k} \) and \( \vec{b} = 3\vec{i} - 6\vec{j} \), express the vector \( 3\vec{a} - 2\vec{b} \) in terms of \( \vec{i}, \vec{j}, \) and \( \vec{k} \).

\[
3\vec{a} - 2\vec{b} = 3(2\vec{i} + \vec{j} - 4\vec{k}) - 2(3\vec{i} - 6\vec{j})
\]

\[
= 6\vec{i} + 3\vec{j} - 12\vec{k} - 6\vec{i} + 12\vec{j}
\]

\[
= 15\vec{j} - 12\vec{k}
\]

* A **unit vector** is a vector whose length is 1. For example, \( \vec{i}, \vec{j}, \) and \( \vec{k} \) are unit vectors. In general, if \( \vec{a} \neq \vec{0} \), then the unit vector that has the same direction as \( \vec{a} \) is

\[
\vec{u} = \frac{1}{|\vec{a}|}\vec{a} = \frac{\vec{a}}{|\vec{a}|}.
\]

**Example:** Find the unit vector in the direction of \( \vec{a} = -2\vec{i} - 3\vec{j} + \vec{k} \).

\[
|\vec{a}| = \sqrt{(-2)^2 + (-3)^2 + 1^2} = (-2, -3, 1)
\]

\[
= \sqrt{14}
\]

Thus, the unit vector is \( \frac{1}{\sqrt{14}}\vec{a} \) is

\[
= \frac{-2}{\sqrt{14}}\vec{i} + \frac{-3}{\sqrt{14}}\vec{j} + \frac{1}{\sqrt{14}}\vec{k}
\]
Example: If \( \vec{v} \) lies in the first quadrant and make an angle of \( \pi/6 \) with the positive \( x \)-axis and \( |\vec{v}| = 3 \), find \( \vec{v} \) in component form.

\[ a = 3 \cdot \cos \frac{\pi}{6} = 3 \cdot \frac{\sqrt{3}}{2} \]
\[ b = 3 \cdot \sin \frac{\pi}{6} = 3 \cdot \frac{1}{2} \]

Thus, \( \vec{v} = \left\langle \frac{3\sqrt{3}}{2}, \frac{3}{2} \right\rangle \)

Example: A cart is pulled along a horizontal path with a force of 60 N exerted at an angle of 25° above the horizontal. Find the horizontal and vertical components of the force.

\[ F_H = 60 \cos 25^\circ \approx 54.4 \text{ N} \]
\[ F_V = 60 \sin 25^\circ \approx 25.4 \text{ N} \]