§13.1 Vector functions and space curves

A **vector-valued function**, (or vector function), is a function whose domain is subset of \( \mathbb{R} \) and whose range is subset of vectors in \( \mathbb{R}^n \).

\[
\vec{r}(t) = \langle f(t), g(t), h(t) \rangle = f(t)\vec{i} + g(t)\vec{j} + h(t)\vec{k} \in \mathbb{R}^3
\]

The functions \( f(t), g(t), h(t) \) are called **component functions**.

**Example.** \( \vec{r}(t) = \langle \sqrt{t-1}, t^2 - t, \ln(4-t) \rangle \)

The component functions are \( f(t) = \sqrt{t-1} \), \( g(t) = t^2 - t \), \( h(t) = \ln(4-t) \)

For the domain, \( t-1 > 0 \) & \( 4-t > 0 \)
\[ t > 1 \text{ and } t < 4 \]

so \( 1 < t < 4 \).

**Definition.** The **limit** of a vector function is defined by taking the limits of its component functions,

\[
\lim_{t \to a} \vec{r}(t) = \left\langle \lim_{t \to a} f(t), \lim_{t \to a} g(t), \lim_{t \to a} h(t) \right\rangle.
\]

A vector function \( \vec{r}(t) \) is **continuous** at \( t = a \) if \( \lim_{t \to a} \vec{r}(t) = \vec{r}(a) \).

**Example.** Find the limit \( \lim_{t \to 0} \vec{r}(t) \) for vector function

\[
\vec{r}(t) = \langle \ln(1-t), t^2 + 2, \frac{\sin t}{t} \rangle.
\]

\[
\lim_{t \to 0} \vec{r}(t) = \left\langle \lim_{t \to 0} \ln(1-t), \lim_{t \to 0} (t^2 + 2), \lim_{t \to 0} \frac{\sin t}{t} \right\rangle
\]

\[
= \left\langle 0, 2, 1 \right\rangle
\]
**Definition.** Suppose \( \vec{r}(t) \) is continuous on an interval \( I \). Then set of points \( (f(t), g(t), h(t)) \) in \( \mathbb{R}^3 \) is called a **space curve**. The equations

\[
x = f(t), \quad y = g(t), \quad z = h(t)
\]

are **parametric equations** of the curve and \( t \) is a **parameter**.

**Example.** Describe the curve defined by the vector function
\[
\vec{r}(t) = (1 + 6t)\hat{i} + (2 - 3t)\hat{j} + (3 - t)\hat{k}.
\]

A line passing through \((1, 2, 3)\) parallel to the vector \(\langle 6, -3, -1 \rangle\). 

\((t=0)\)

**Example.** Find a vector equation and parametric equations for the line segment that joins the points \(P(3, -2, 4)\) and \(Q(-1, 2, 4)\)

**Direction vector**

\[
\vec{v} = \vec{PQ} = \langle -4, 4, 0 \rangle = \vec{r}_1 - \vec{r}_0
\]

\[
\vec{r}_0 = \langle 3, -2, 4 \rangle \quad \vec{r}_1 = \langle -1, 2, 4 \rangle
\]

**Vector equation**

\[
\vec{r} = \vec{r}_0 + t \vec{v}
\]

\[
= \vec{r}_0 + t(\vec{r}_1 - \vec{r}_0)
\]

\[
= (1-t)\vec{r}_0 + t\vec{r}_1
\]

\[
= (1-t)\langle 3, -2, 4 \rangle + t\langle -1, 2, 4 \rangle
\]

\[
= \langle 3 - 4t, 2 + 4t, 4 \rangle
\]

**Parametric Equation**

\[
\begin{align*}
x &= 3 - 4t \\
y &= 2 + 4t \\
z &= 4
\end{align*}
\]
Example. [Helix] Sketch the curve whose vector equation is \( \vec{r}(t) = (\cos t, \sin t, t) \).

- parametric equation \( X = \cos t, \ Y = \sin t, \ Z = t \)
- projection on \( xy \)-plane is \( (\cos t, \sin t, 0) \), a circle.

Example. Find a vector function for the curve of intersection of the cylinder \( x^2 + y^2 = 4 \) and the plane \( y + z = 3 \).

\[
\frac{x^2}{2^2} + \frac{y^2}{2^2} = 1
\]
Let \( \frac{x}{2} = \sin t \), \( \frac{y}{2} = \cos t \), \( 0 \leq t \leq 2\pi \)

\( \Rightarrow \) \( x = 2 \sin t \), \( y = 2 \cos t \)

Substitute to the plane \( y + z = 3 \)

\( 2 \cos t + z = 3 \)

\( \Rightarrow \) \( z = 3 - 2 \cos t \)

The intersection has parametric equation

\[
\begin{align*}
X &= 2 \sin t \\
Y &= 2 \cos t \\
Z &= 3 - 2 \cos t
\end{align*}
\]

The vector equation is

\[
\vec{r}(t) = (2 \sin t) \mathbf{i} + (2 \cos t) \mathbf{j} + (3 - 2 \cos t) \mathbf{k}
\]

parametrization of the curve.
Example. Find the projection of the space curve $\vec{r}(x) = (t, 2t, 3t + 2t^2)$ onto the coordinate planes.

The projection onto $xy$-plane is $<t, 2t, 0>$

$xz$-plane is $<t, 0, 3t+2t^2>$

$yz$-plane is $<0, 2t, 3t+2t^2>$

Example. Find the intersection points of the space curve $\vec{r}(x) = (t, 2t, 3t + 2t^2)$ and the paraboloid $z = x^2 + y^2$.

Substitute $x=t, y=2t, z=3t+2t^2$ to the paraboloid $z = x^2 + y^2$

$3t+2t^2 = t^2 + (2t)^2$

$3t = 3t^2$

$t(t-1) = 0$

$t=0$ or $t=1$

For $t=0$ \[ \begin{cases} x=0 \\ y=0 \\ z=0 \end{cases} \]

For $t=1$ \[ \begin{cases} x=1 \\ y=2 \\ z=5 \end{cases} \]

There are two intersection points $(0,0,0)$ and $(1,2,5)$