§14.2 Limits and Continuity

Definition. Let $f$ be a function of two variables whose domain $D$ includes points arbitrarily close to $(a, b)$. Then we say that the limit of $f(x, y)$ as $(x, y)$ approaches $(a, b)$ is $L$ and we write

$$\lim_{(x,y)\to(a,b)} f(x, y) = L$$

if for every number $\epsilon > 0$, there is a corresponding number $\delta(\epsilon) > 0$ such that if $(x, y) \in D$ and $0 < \text{dist}((x, y), (a, b)) < \delta$ then

$$|f(x, y) - L| < \epsilon.$$

Here, $\text{dist}((x, y), (a, b)) = \sqrt{(x - a)^2 + (y - b)^2}$.

Example. $\lim_{(x,y)\to(0,0)} e^{-xy} \sin(x + y) = 0 = f(0, 0)$

Example. $\lim_{(x,y)\to(1,2)} \frac{x + y^2}{x^2 - y^2} = \frac{5}{-3} = f(1, 2)$

Definition. A function $f$ of two variables is called continuous at $(a, b)$ if

$$\lim_{(x,y)\to(a,b)} f(x, y) = f(a, b)$$
**Example.** Compute \( \lim_{(x,y) \to (0,0)} f(x, y) \) and \( \lim_{(x,y) \to (1,1)} f(x, y) \) for \( f(x, y) = \frac{x^4 - y^4}{x - y} \):

\[
f(x, y) = \frac{(x^4 - y^4)(x^4 + y^4)}{x - y}
\]

\[
= \frac{(x-y)(x+y)(x^4 + y^4)}{x-y}
\]

\[
= (x+y)(x^4 + y^4)
\]

\[
\lim_{(x,y) \to (0,0)} f(x,y) = 0
\]

\[
\lim_{(x,y) \to (1,1)} f(x,y) = 2 \times 2 = 4
\]

**Example.** Compute \( \lim_{(x,y) \to (0,0)} f(x, y) \) for \( f(x, y) = \frac{x^2 - y^2}{x^2 + y^2} \):

- Approach \((0,0)\) along \(x\)-axis \((y=0)\)
  \[
f(x, 0) = \frac{x^2}{x^2} = 1 \quad \text{for all } x \neq 0.
\]

- Approach \((0,0)\) along \(y\)-axis \((x=0)\)
  \[
f(0, y) = \frac{-y^2}{y^2} = -1 \quad \text{for all } y \neq 0.
\]

\(f\) has different limits along two different lines.

Then, \( \lim_{(x,y) \to (0,0)} f(x, y) \) does not exist.
Example. Compute \( \lim_{(x,y) \to (0,0)} f(x, y) \) for \( f(x, y) = \frac{xy^2}{x^2 + y^4} \)

- **Approach \((0, 0)\) along \(-\text{axis} \) \((y=0)\)
  \[ f(x, 0) = 0 \quad \text{for} \quad x \neq 0. \]

- **Approach \((0, 0)\) along \(y\)-axis \((x=0)\)
  \[ f(0, y) = 0 \quad \text{for} \quad y \neq 0. \]

- **Approach \((0, 0)\) along \(x = y^2\)
  \[ f(x, y) = \frac{y^2y^2}{y^4 + y^4} = \frac{1}{2} \]

So, \( \lim_{(x,y) \to (0,0)} f(x, y) \) does not exist.

Example: \( f(x, y) = \frac{\sin(x^2 + y^2)}{x^2 + y^2} \)

\[ = \frac{\sin t}{t} \quad \text{where} \quad t = x^2 + y^2 > 0 \]

\[ \lim_{(x, y) \to (0, 0)} f(x, y) = \lim_{t \to 0^+} \frac{\sin t}{t} = 1 \quad \text{L’Hospital’s Rule}. \]