• **Review**: Single integral over interval.

1. **Definition.** The **definite integral** of a continuous function $f(x)$ on $[a, b]$ is defined as the limit of Riemann sum:

\[
\int_a^b f(x)\,dx = \lim_{\Delta x \to 0} \sum_{i} f(x_i^*)\Delta x.
\]

2. **Estimation.** Definite integral of $f(x)$ from $a$ to $b$ can be estimated by Riemann sum $\int_a^b f(x)\,dx \approx \sum_{i=1}^{n} f(x_i^*)\Delta x$.

3. **Calculation.** (The Fundamental Theorem of Calculus.)

If $F(x)$ is any anti-derivative of $f(x)$, then

\[
\int_a^b f(x)\,dx = F(x)\bigg|_a^b = F(b) - F(a).
\]

Example. \[
\int_1^7 x^2 \,dx = \frac{x^3}{3}\bigg|_1^7 = \frac{7^3}{3} - \frac{1^3}{3} = 114.
\]
• Double integral of $f(x, y)$ on region $R$

1. Definition. Riemann integral (concepts)

The **double integral** of $f(x, y)$ on $R$ is defined as limits of Riemann sum:

$$\int \int_R f(x, y) \, dA = \lim_{\Delta A \to 0} \sum_{i,j} f(x_{ij}^*, y_{ij}^*) \Delta A.$$ 

Here, $\Delta A = (\Delta x)(\Delta y)$. 

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[Diagram showing a 3D representation of the double integral over a region $R$]
Geometric meaning:
If \( f(x, y) \) is positive, then \( \iint_R f(x, y) \, dA \) is the volume of the columns cylinder between \( f(x, y) \) and \( xy \)-plane on area \( R \).

**Example.** Evaluate the double integral by first identifying it as the volume of a solid.

\[
\iint_R \sqrt{1-x^2} \, dA \\
R = \{ (x, y) \mid 0 \leq x \leq 1, \ 0 \leq y \leq 2 \}
\]

\[
\cdot \quad z = f(x, y) = \sqrt{1-x^2} \quad \Rightarrow \quad x^2+z^2=1, \ \text{graph cylinder}
\]

The volume of the cylinder on \( R \) is

\[
V = \frac{1}{4} (\pi r^2) \cdot h = \frac{1}{4} \cdot (\pi) \cdot 2 = \frac{\pi}{2}
\]

\[
\iint_R \sqrt{1-x^2} \, dA = \frac{\pi}{2}
\]
2. Estimations:

Example. Estimate the volume of the solid that lies below the surface

$$z = x^2y$$

and above the following rectangle

$$R = \{(x, y) \in \mathbb{R}^2 \mid 1 \leq x \leq 4 \text{ and } 0 \leq y \leq 2\}$$

Use a Riemann sum with $m = 3, n = 2$, and take the sample point to be the upper right corner of each square.

$$\sqrt{\sum \sum f(x_i, y_j) \Delta A}$$

$$= f(2, 1) \Delta A + f(2, 2) \Delta A + f(3, 1) \Delta A + f(3, 2) \Delta A$$

$$+ f(4, 1) \Delta A + f(4, 2) \Delta A$$

$$= 4x^1 + 8x^1 + 9x^1 + 18x^1 + 16x^1 + 32x^1$$

$$= 87$$
**The Midpoint Rule:** The double integral of \( f(x, y) \) on \( R \) can be estimated as

\[
\iint_R f(x, y) \, dA \approx \sum_{i=1}^{m} \sum_{j=1}^{n} f(\bar{x}_i, \bar{y}_i) \Delta A.
\]

Here, \( \bar{x}_i \) is the midpoint of \([x_{i-1}, x_i]\) and \( \bar{y}_i \) is the midpoint of \([y_{i-1}, y_i]\).

**Example.** Use a Midpoint Rule with \( m = 3, n = 2 \). Estimate the double integral of \( f(x, y) = x^2y \) on the rectangle \( R = \{(x, y) \in \mathbb{R}^2 \mid 1 \leq x \leq 4 \text{ and } 0 \leq y \leq 2\} \)

\[
V \approx \sum_{i=1}^{3} \sum_{j=1}^{2} f(\bar{x}_i, \bar{y}_i) \Delta A
\]

\[
= f(1.5, 0.5) \Delta A + f(1.5, 1.5) \Delta A + f(2.5, 0.5) \Delta A
\]

\[
+ f(2.5, 1.5) \Delta A + f(3.5, 0.5) \Delta A + f(3.5, 1.5) \Delta A
\]

\[
= 41.5
\]
3. Calculations: (Fubini’s Theorem)
Suppose \( f = f(x, y) \) is continuous on the rectangle \( R = [a, b] \times [c, d] \).

- Similarly as partial derivative, we can calculate **Partial Integral**
  \[
  \int_{a}^{b} f(x, y) \, dx
  \]
  respect to \( x \) by thinking \( y \) as constant.

**Example.** \( f(x, y) = x^2 y \) for \( 1 \leq x \leq 4 \) and \( 0 \leq y \leq 2 \).

\[
\int_{a}^{b} f(x, y) \, dx = \int_{1}^{4} x^2 y \, dx = \frac{x^3}{3} \Bigg|_{x=1}^{4} = \frac{64}{3} - \frac{1}{3} y = 21 y
\]

The **iterated integral** is
\[
\int_{c}^{d} \left( \int_{a}^{b} f(x, y) \, dx \right) \, dy
\]

**Example.**

\[
\int_{c}^{d} \int_{a}^{b} f(x, y) \, dx \, dy = \int_{0}^{2} \left( \int_{1}^{4} x^2 y \, dx \right) \, dy = \int_{0}^{2} 21 y \, dy = \frac{21 y^2}{2} \Bigg|_{0}^{2} = 42
\]

**Fubini’s Theorem.**
The double integral can be calculated by iterated integrals:
\[
\iint_{R} f(x, y) \, dA = \int_{a}^{b} \int_{c}^{d} f(x, y) \, dy \, dx = \int_{c}^{d} \int_{a}^{b} f(x, y) \, dx \, dy
\]

**Example.**

\[
\iint_{R} f(x, y) \, dA = \int_{0}^{2} \int_{1}^{4} x^2 y \, dx \, dy = 42
\]
Example. Calculate the double integral of \( f(x, y) = x^2 y \) on the region \( R = \{(x, y) \in \mathbb{R}^2 \mid 1 \leq x \leq 4 \text{ and } 0 \leq y \leq 2\} \)

\[
\iint_R f(x, y) \, dA = \int_1^4 \left( \int_0^2 x^2 y \, dy \right) \, dx
\]

\[
= \int_1^4 \left( \frac{x^2 y^2}{2} \bigg|_0^2 \right) \, dx
\]

\[
= \int_1^4 2x^2 \, dx = \frac{2x^3}{3}\bigg|_1^4 = 42
\]

Example. The region \( R \) is given by \( 0 \leq x \leq 2 \) and \( 1 \leq y \leq 3 \). Calculate the double integral \( \iint_R f(x, y) \, dA \) for \( f(x, y) = 1 - x^2 - y^2 \).

\[
\iint_R f(x, y) \, dA = \int_1^3 \int_1^2 1 - x^2 - y^2 \, dx \, dy
\]

\[
= \int_1^3 \left( x - \frac{x^2}{2} - xy^2 \bigg|_0 \right) \, dy
\]

\[
= \int_1^3 \left( \left(x - \frac{x^2}{2} - xy^2 \right) \bigg|_0 \right) \, dy
\]

\[
= \int_1^3 \frac{-2y^2}{3} \, dy
\]

\[
= \left[-\frac{2}{3}y - \frac{2}{3}y^3\right]_1^3 = -\frac{56}{3}
\]
**Example.** Calculate the iterated integral \( \int_{-1}^{1} \int_{0}^{\pi/2} f(x, y) \, dx \, dy \) for 
\[ f(x, y) = 2y + y^3 \cos x. \]

\[
\int_{0}^{\pi/2} 2y + y^3 \cos x \, dx = 2yx + y^3 \sin x \bigg|_{0}^{\pi/2} = \pi y + y^3
\]

\[
\int_{-1}^{1} \int_{0}^{\pi/2} f(x, y) \, dx \, dy = \int_{-1}^{1} (\pi y + y^3) \, dy
\]

\[
= \frac{\pi y^2}{2} + \frac{y^4}{4} \bigg|_{-1}^{1} = 0
\]

**Example.** Calculate the iterated integral

\[
\int_{0}^{2} \int_{0}^{3} 2e^{x+2y} \, dx \, dy
\]

Method 2:

\[
\int_{0}^{2} \int_{0}^{3} 2e^{x+2y} \, dx \, dy = \int_{0}^{2} \int_{0}^{3} 2e^{x+2y} \, dx \, dy
\]

\[
= \int_{0}^{2} \left( \int_{0}^{3} 2e^{x+2y} \, dx \right) \, dy
\]

\[
= \int_{0}^{2} \left[ 2e^{x+2y} \bigg|_{0}^{3} \right] \, dy
\]

\[
= \int_{0}^{2} \left( 2e^{5+2y} - 2e^{2+2y} \right) \, dy
\]

\[
= \int_{0}^{2} e^{2y+3} \, dy - \int_{0}^{2} e^{2y} \, dy
\]

\[
= e^{2y+3} \bigg|_{0}^{2} - e^{2y} \bigg|_{0}^{2} = e^{8} - e^{2} - (e^{4} - 1)
\]

\[
= e^{7} - e^{3} - e^{4} + 1
\]
**Average Value**

The average value of a function $f(x, y)$ over a rectangle $R$ is defined to be

$$f_{avg} = \frac{1}{A(R)} \iiint_R f(x, y) \, dA$$

where $A(R)$ is the area of $R$.

**Example** Find the average value of the paraboloid $f(x, y) = x^2 + y^2$ over $R = [-1, 1] \times [-1, 1]$.

\[
\begin{align*}
    f_{avg} &= \frac{1}{A(R)} \int_{-1}^{1} \int_{-1}^{1} (x^2 + y^2) \, dx \, dy \\
    &= \frac{1}{4} \int_{-1}^{1} \left[ \frac{2}{3}x^3 + \frac{2}{3}y^3 \right] \, dy \\
    &= \frac{1}{4} \left[ \left( \frac{2}{3}x^3 + \frac{2}{3}y^3 \right) \right]_{-1}^{1} \\
    &= \frac{1}{4} \left( \frac{4}{3} + \frac{4}{3} \right) \\
    &= \frac{2}{3} 
\end{align*}
\]