Line Integral plane in \( \mathbb{R}^2 \)

Recall: §13.3 Suppose a smooth curve \( C \) has the vector equation \( \vec{r}(t) = (x(t), y(t)) \) for \( a \leq t \leq b \). If the curve is traversed exactly once as increases from \( a \) to \( b \), then its length is

\[
L = \int_a^b |\vec{r}'(t)| \, dt = \int_a^b \sqrt{|x'(t)|^2 + |y'(t)|^2} \, dt = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dt
\]

**Definition.** If \( f \) is a function defined on \( C \), then the line integral of \( f \) along \( C \) is

\[
\int_C f(x, y) \, ds = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*, y_i^*) \Delta s_i
\]

**Computation.** The line integral of \( f \) along \( C \) can be evaluated as

\[
\int_C f(x, y) \, ds = \int_a^b f(x(t), y(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dt
\]

Recall: The arc length function \( s(t) \) is the length of the curve between \( \vec{r}(a) \) and \( \vec{r}(t) \) defined by \( s(t) = \int_a^t |\vec{r}'(u)| \, du = \int_a^t \sqrt{\left(\frac{dx}{du}\right)^2 + \left(\frac{dy}{du}\right)^2} \, du \)

From the Fundamental Theorem of Calculus, differentiate both sides, we have

\[
\frac{ds}{dt} = |\vec{r}'(t)| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}
\]
Example 1. Evaluate $\int_C (3 - xy^2)\,ds$, where $C$ is the first quadrant of the unit circle $x^2 + y^2 = 1$.

\[
X = \cos t \quad x'(t) = -\sin t \\
y = \sin t \quad y'(t) = \cos t \\
0 \leq t \leq \frac{\pi}{2}
\]

\[
\int_C 3 - xy^2 \,ds = \int_0^{\frac{\pi}{2}} \left[ 3 - (\cos t)(\sin^2 t) \right] \sqrt{\left(x'(t)\right)^2 + \left(y'(t)\right)^2} \, dt
\]

\[
= \int_0^{\frac{\pi}{2}} (3 - \sin^2 t \cos t) \, dt
\]

\[
= 3t - \frac{\sin^3 t}{3} \bigg|_0^{\frac{\pi}{2}}
\]

\[
= \frac{3\pi}{2} - \frac{1}{3}
\]

Let $\rho(x, y)$ be the density function on a curve (wire) $C$. Then the mass of the wire $C$ is

\[
m = \lim_{n \to \infty} \sum_{i=1}^{n} \rho(x_i^*, y_i^*) \Delta s_i = \int_C \rho(x, y)\,ds
\]

The center of mass is $(\bar{x}, \bar{y})$ computed by

\[
\bar{x} = \frac{1}{m} \int_C x\rho(x, y)\,ds \quad \bar{y} = \frac{1}{m} \int_C y\rho(x, y)\,ds
\]
Suppose \( C \) is a piecewise-smooth curve.

Then,

\[
\int_C f(x, y) ds = \int_{C_1} f(x, y) ds + \int_{C_2} f(x, y) ds + \cdots + \int_{C_n} f(x, y) ds
\]

**Example 2.** Evaluate \( \int_C 2x \, ds \), where \( C \) is the arc \( C_1 \) of the parabola \( y = x^2 \) from \((0, 0)\) to \((1, 1)\) followed by the line segment \( C_2 \) from \((1, 1)\) to \((2, 1)\).

\[
\int_C 2x \, ds = \int_{C_1} 2x \, ds + \int_{C_2} 2x \, ds
\]

1. \( C_1 \), \( x = t \), \( y = t^2 \)
   \[
   \int_{C_1} 2x \, ds = \int_0^1 2t \sqrt{1 + (2t)^2} \, dt
   = \int_0^1 2t \sqrt{1 + 4t^2} \, dt
   = \int_0^1 (1 + 4t^2)^{1/2} \, dt
   = \frac{1}{4} \cdot \frac{2}{3} (1 + 4t^2)^{3/2} \bigg|_0^1 = \frac{5\sqrt{5} - 1}{6}
   \]

2. \( C_2 \), \( x = t \), \( y = 1 \)
   \[
   \int_{C_2} 2x \, ds = \int_1^2 2t \sqrt{1 + 0} \, dt
   = t^2 \bigg|_1^2 = 4 - 1 = 3
   \]

So,
\[
\int_C 2x \, ds = \frac{5\sqrt{5} - 1}{6} + 3
\]
The line integral of $f$ along $C$ with respect to $x$ is
\[ \int_C f(x, y)dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*, y_i^*) \Delta x_i \]

The line integral of $f$ along $C$ with respect to $y$ is
\[ \int_C f(x, y)dy = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*, y_i^*) \Delta y_i \]

Suppose a smooth curve $C$ has the vector equation $\vec{r}(t) = (x(t), y(t))$ for $a \leq t \leq b$. The line integral of $f$ along $C$ with respect to $x$ and $y$ can be evaluated as
\[ \int_C f(x, y)dx = \int_a^b f(x(t), y(t))x'(t)dt \]
\[ \int_C f(x, y)dy = \int_a^b f(x(t), y(t))y'(t)dt \]

$\int_C f(x, y)ds$ will be called the line integral of $f$ along $C$ with respect to arc length.

Notation:
\[ \int_C f(x, y)dx + g(x, y)dy := \int_C f(x, y)dx + \int_C g(x, y)dy \]
**Example 3.** Evaluate \( \int_C y^2 \, dx - 2x \, dy \), where \( C \) is the line segment from \((-4, -2)\) to \((1, 2)\).

Position vector \( \mathbf{r}_o = \langle -4, -2 \rangle \)

Direction vector \( \mathbf{v} = \mathbf{PQ} = \langle 5, 4 \rangle \)

**Line Segment \( C \):**

\[ \mathbf{r}(t) = \mathbf{r}_o + t \mathbf{v} = \langle -4, -2 \rangle + t \langle 5, 4 \rangle \]

\[ x = -4 + 5t \quad 0 \leq t \leq 1 \]

\[ y = -2 + 4t \]

\[ dx = 5 \, dt \]

\[ dy = 4 \, dt \]

\[
\int_C y^2 \, dx - 2x \, dy = \int_0^1 (2+4t)^2 \cdot 5 \, dt - 2(-4+5t) \cdot 4 \, dt
\]

\[
= \int_0^1 5(16t^2 - 24t + 4) + 32 \, dt
\]

\[
= 5 \left( \frac{16}{3} t^3 - 12t^2 + 4t \right) + 32t \bigg|_0^1
\]

\[
= \frac{56}{3}
\]
Example 4. Evaluate $\int_C y^2 \, dx$, where $C$ is the arc of the parabola $x = 2 - y^2$ from $(1, -1)$ to $(-2, 2)$.

$x = 2 - t^2$

$y = t$

$-1 \leq t \leq 2$

$$\int_C y^2 \, dx = \int_{-1}^{2} t^2 \, x(t) \, dt$$

$$= \int_{-1}^{2} t^2 \, (-2t) \, dt$$

$$= \int_{-1}^{2} -2 t^3 \, dt$$

$$= -2 \left. \frac{t^4}{4} \right|_{-1}^{2}$$

$$= - \frac{15}{2}$$
**Line integral in space \( \mathbb{R}^3 \).**
Suppose a smooth curve \( C \) has the vector equation \( \mathbf{r}'(t) = (x(t), y(t), z(t)) \) for \( a \leq t \leq b \).

**Definition.** The line integral of \( f \) along \( C \) **with respect to the arc length** is

\[
\int_C f(x, y, z) \, ds = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*, y_i^*, z_i^*) \Delta s_i
\]

The line integral of \( f \) along \( C \) **with respect to** \( z \) is

\[
\int_C f(x, y, z) \, dz = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*, y_i^*, z_i^*) \Delta z_i
\]

**Computation.** The line integral of \( f \) along \( C \) with respect to the arc length can be evaluated as

\[
\int_C f(x, y, z) \, ds = \int_a^b f(x(t), y(t), z(t)) \sqrt{\left( \frac{dx}{dt} \right)^2 + \left( \frac{dy}{dt} \right)^2 + \left( \frac{dz}{dt} \right)^2} \, dt
\]

The line integral of \( f \) along \( C \) with respect to \( z \) can be evaluated as

\[
\int_C f(x, y, z) \, dz = \int_a^b f(x(t), y(t), z(t)) z'(t) \, dt
\]
**Example 5.** Evaluate $\int_C 2x \sin z \, ds$, where $C$ is the helix defined by $x = \sin t$, $y = \cos t$, $z = t$ for $0 \leq t \leq \pi$.

\[
\int_C 2x \sin z \, ds = \int_0^\pi 2 \sin t \sin t \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2} \, dt
\]

\[
= \int_0^\pi 2 \sin^2 t \sqrt{\cos^2 t + \sin^2 t + 1} \, dt
\]

\[
= \int_0^\pi 2 \sin^2 t \, dt
\]

\[
= \sqrt{2} \int_0^\pi 1 - \cos 2t \, dt
\]

\[
= \sqrt{2} \left( t - \frac{1}{2} \sin 2t \right) \bigg|_0^\pi
\]

\[
= \sqrt{2} \pi
\]
Example 6. Evaluate $\int_C y\,dx + z\,dy + x\,dz$, where $C$ is the union of the line segment $C_1$ from $(3, 4, 0)$ to $(3, 4, 5)$ and the line segment $C_2$ from $(3, 4, 5)$ to $(2, 0, 0)$.

For $C_1$: \[ \mathbf{r}(t) = \langle 3, 4, 0 \rangle + t \langle 0, 0, 5 \rangle \]
\[ x = 3 \quad y = 4 \quad z = 5t \quad 0 \leq t \leq 1 \]
\[ \int_{C_1} y\,dx + z\,dy + x\,dz = \int_0^1 (4 - 4t)(0) + (5t)(0) + (3)(5) \, dt \]
\[ = 15 \]

For $C_2$: \[ \mathbf{r}(t) = \langle 3, 4, 5 \rangle + t \langle -1, -4, -5 \rangle \]
\[ x = 3 - t \quad y = 4 - 4t \quad z = 5 - 5t \quad 0 \leq t \leq 1 \]
\[ \int_{C_2} y\,dx + z\,dy + x\,dz = \int_0^1 (4 - 4t)(-1) + (5 - 5t)(-4) + (3 - t)(-5) \, dt \]
\[ = \int_0^1 29t - 39 \, dt \]
\[ = \left. \frac{29t^2}{2} - 39t \right|_0^1 = -24.5 \]

So, \[ \int_C y\,dx + z\,dy + x\,dz = 15 - 24.5 = -9.5 \]
Line Integrals of Vector Fields.

**Calculus 1.** The work done by a force function \( f(x) \) in moving a particle from \( a \) to \( b \) along \( x \)-axis is \( W = \int_a^b f(x) \, dx \).

**§12.3.** The work done by a constant force \( \vec{F} \) along displacement vector \( \vec{D} \) is given by \( W = \vec{F} \cdot \vec{D} \).

**Question:** How to calculate the work done by a force function \( \vec{F}(x, y, z) \) moving a particle along a curve \( C \)?

\[
\sum_{i=1}^{n} \vec{F}(x_i^*, y_i^*, z_i^*) \cdot [\Delta S_i \cdot \vec{T}(t_i^*)] = \int_C \vec{F} \cdot \vec{T} \, ds
\]
**Definition.** Let \( \vec{F} \) be a vector field (on \( \mathbb{R}^2 \) or \( \mathbb{R}^3 \)) defined on a curve \( C (\vec{r}(t), a \leq t \leq b) \). Then the line integral of \( \vec{F} \) along \( C \) is

\[
\int_C \vec{F} \cdot \vec{T} \, ds = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) \, dt = \int_C \vec{F} \cdot d\vec{r}
\]

where \( \vec{T} \) is the tangent vector at the point \((x, y, z) \in C\).

\[
\vec{T}(t) = \frac{\vec{r}'(t)}{|\vec{r}'(t)|}, \quad ds = |\vec{r}'(t)| \, dt
\]

\[
\int_C \vec{F} \cdot \vec{T} \, ds = \int_C \vec{F} \cdot \frac{\vec{r}'(t)}{|\vec{r}'(t)|} \cdot |\vec{r}'(t)| \, dt
\]

\[
= \int_C \vec{F} \cdot \vec{r}'(t) \, dt.
\]

- If \( \vec{F} = \langle P, Q, R \rangle \), \( \vec{r}(t) = \langle x(t), y(t), z(t) \rangle \)

  then

  \[
  \int_C \vec{F} \cdot \vec{T} \, ds = \int_a^b \vec{F} \cdot \vec{r}'(t) \, dt
  \]

  \[
  = \int_a^b (P \, x'(t) + Q \, y'(t) + R \, z'(t)) \, dt.
  \]

  \[
  = \int_C P \, dx + Q \, dy + R \, dz.
  \]
Example 7. Find the work done by a force field \( \vec{F}(x, y) = \langle y^2, -xy \rangle \) moving a particle along the curve \( C \) given by \( \vec{r}(t) = \langle \sin t, \cos t \rangle \), when \( 0 \leq t \leq \pi/2 \).

\[
\int_C \vec{F} \cdot d\vec{r} = \int_0^{\pi/2} \langle y^2, -xy \rangle \cdot \vec{r}'(t) \, dt
\]

\[
= \int_0^{\pi/2} \langle \cos^2 t, -\sin t \cos t \rangle \langle \cos t, -\sin t \rangle \, dt
\]

\[
= \int_0^{\pi/2} \cos^3 t + \sin^2 t \cos t \, dt
\]

\[
= \int_0^{\pi/2} \cos t \, dt
\]

\[
= \left. \sin t \right|_0^{\pi/2}
\]

\[
= 1
\]
Example 8. Evaluate $\int_C \vec{F} \cdot d\vec{r}$, where $\vec{F}(x, y, z) = \langle xy, yz, zx \rangle$ and $C$ is given by $x = t$, $y = t^2$, $z = t^3$ for $0 \leq t \leq 1$.

$$\int_C \vec{F} \cdot d\vec{r} = \int_{\theta}^{1} \vec{F} \cdot \vec{r}'(t) \, dt$$

$$= \int_{0}^{1} \langle t^3, t^5, t^4 \rangle \cdot \langle 1, 2t, 3t^2 \rangle \, dt$$

$$= \int_{0}^{1} t^3 + 5t^6 \, dt$$

$$= \left. \frac{t^4}{4} + \frac{5t^7}{7} \right|_0^1$$

$$= \frac{27}{28}$$