§16.3 The Fundamental Theorem for Line Integrals

Review of the Fundamental Theorem of Calculus:

If $F'(x)$ is continuous on the interval $[a, b]$, then

$$\int_{a}^{b} F'(x)\,dx = F(b) - F(a)$$

Let $C$ be a curve defined vector function $\vec{r}(t)$, $a \leq t \leq b$.
Let $\vec{F}$ be a vector field (on $\mathbb{R}^2$ or $\mathbb{R}^3$) defined on $C$.
Then the line integral of $\vec{F}$ along $C$ is defined as $\int_C \vec{F} \cdot d\vec{r}$.

The Fundamental Theorem for line integrals.

**Theorem.** If $C$ is smooth and $\vec{F}$ is conservative ($\vec{F} = \nabla f$), then

$$\int_C \vec{F} \cdot d\vec{r} = \int_C \nabla f \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a)).$$

This means that, in this nice situation, we can evaluate the line integral using only value at the endpoints of the potential function $f$.

**Proof:**

$$\int_C \nabla f \cdot d\vec{r} = \int_a^b \nabla f \cdot r'(t) \, dt$$

$$= \int_a^b \langle f_x, f_y \rangle \cdot \langle x'(t), y'(t) \rangle \, dt$$

$$= \int_a^b f_x \cdot x'(t) + f_y \cdot y'(t) \, dt$$

$$= \int_a^b \frac{d}{dt}(f(\vec{r}(t))) \, dt$$

$$= f(\vec{r}(b)) - f(\vec{r}(a))$$
If $\vec{F} = \nabla f$ is defined on a region $D$ and $C_1$, $C_2$ are two (piecewise) smooth curves on $D$ with the same initial and terminal points, then

$$\int_{C_1} \vec{F} \cdot d\vec{r} = \int_{C_2} \vec{F} \cdot d\vec{r}$$

Line integrals of conservative vector fields are independent of path.

$$\vec{F}(x, y) = \langle y^4 - 3x^2, 2xy \rangle$$

$$f(x, y) = y^2x - x^3$$

$$\vec{F} = \nabla f$$

$A = (2, 1)$

$B = (7, 8)$

$$\int_C \vec{F} \cdot d\vec{r} = f(7, 8) - f(2, 1) = 105 - 6 = 99$$

for any $C$ from $A$ to $B$. 

for any $C$ from $A$ to $B$. 

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Example 1. (Gravitational Field) Let $\vec{x} = (x, y, z) \in \mathbb{R}^3$. The gravitational force acting on the object at $\vec{x}$ is

$$\vec{F}(\vec{x}) = -\frac{mMG}{|\vec{x}|^3} \vec{x}$$

$m$ and $M$ are masses of the two objects. $G = 6.67408 \times 10^{-11}$ is the universal Gravitational constant.

The Gravitational Field is a conservative vector field, $\vec{F} = \nabla f$, for

$$f(x, y, z) = \frac{mMG}{\sqrt{x^2 + y^2 + z^2}}$$

**Problem:** Find the work done by $\vec{F}$ in moving a particle with mass $m$ from point $(1, 0, 0)$ to $(1, 2, 9)$.

By fundamental theorem for line integral

$$\text{work} = \int_C \vec{F} \cdot d\vec{r} = \int_C \nabla f \cdot d\vec{r} = f(1, 2, 9) - f(1, 0, 0)$$

$$= \frac{mMG}{\sqrt{1+4+81}} - \frac{mMG}{\sqrt{1}}$$

$$= \left( \frac{1}{\sqrt{86}} - 1 \right) mMG$$
A curve is **closed** if its terminal point coincides with its initial point.

**Theorem.** The line integral $\int_C \vec{F} \cdot d\vec{r}$ is independent of path in $D$ if and only if $\int_C \vec{F} \cdot d\vec{r} = 0$ for every closed path $C$ in $D$.

\[
\int_C \vec{F} \cdot d\vec{r} = \int_{C_1} \vec{F} \cdot d\vec{r} + \int_{C_2} \vec{F} \cdot d\vec{r}
\]

\[
= \int_{C_1} \vec{F} \cdot d\vec{r} - \int_{C_2} \vec{F} \cdot d\vec{r}
\]

\[
= 0
\]

Suppose $\vec{F}$ is continuous on an **open, connected** region $D$.

**Theorem.** The line integral $\int_C \vec{F} \cdot d\vec{r}$ is independent of path in $D$ if and only if $\vec{F}$ is conservative on $D$, that is, $\vec{F} = \nabla f$.

**Question:**

How to determine whether or not a vector field $\vec{F}$ is conservative?

**Theorem.** If $\vec{F}(x, y) = P(x, y)i + Q(x, y)j$ is conservative on $D$, then

\[
\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}
\]
Theorem. Let $D$ be a simply-connected open region. Let $\vec{F}(x, y) = P(x, y)\vec{i} + Q(x, y)\vec{j}$ be a vector field such that $P$ and $Q$ have continuous first derivatives and
\[ \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \]
on $D$, then $\vec{F}$ is conservative.

A region $D$ in the plane $\mathbb{R}$ is simply-connected if it is connected and every simple closed curve in $D$ enclose only points in $D$. 
Example 2. Determine whether or not each of the vector field is conservative.

1. \( \vec{F}(x, y) = (2x + y)\hat{i} + (x + 2y)\hat{j} \) is continuous on \( \mathbb{R}^2 \).
   \[ \frac{\partial P}{\partial y} = 1 \quad \Rightarrow \quad \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \quad \Rightarrow \quad \text{conservative} \]
   \[ \frac{\partial Q}{\partial x} = 1 \]

2. \( \vec{F}(x, y) = (2x - y, x + 1) \)
   \[ \frac{\partial P}{\partial y} = -1 \quad \Rightarrow \quad \frac{\partial P}{\partial y} \neq \frac{\partial Q}{\partial x} \quad \Rightarrow \quad \text{Not conservative} \]
   \[ \frac{\partial Q}{\partial x} = 1 \]

3. \( \vec{F}(x, y) = (4 + 2xy, x^2 + y^2) \)
   \[ \frac{\partial P}{\partial y} = 2x \quad \Rightarrow \quad \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \quad \Rightarrow \quad \text{conservative} \]
Example 3. Let \( \vec{F}(x, y) = \langle 4 + 2xy, x^2 + y^2 \rangle \).

(1) Find a function \( f \) such that \( \nabla f = \vec{F} \).

\[
\nabla f = \langle f_x , f_y \rangle = \vec{F} = \langle 4 + 2xy, x^2 + y^2 \rangle
\]

So \( f_x = 4 + 2xy \)

\( f_y = x^2 + y^2 \)

\[
\Rightarrow x^2 + y^2 = x^2 + g'(y) \Rightarrow g'(y) = y^2 \Rightarrow g(y) = \frac{y^3}{3} + k
\]

So \( f(x, y) = 4x + x^2y + \frac{y^3}{3} + k \)

(2) Evaluate the line integral \( \int_C \vec{F} \cdot d\vec{r} \), where \( C \) is the curve defined by \( \vec{r}(t) = \langle e^t \cos t, e^t \sin t \rangle \) for \( 0 \leq t \leq \pi/2 \).

\[
\vec{r}(0) = \langle 1, 0 \rangle \quad \text{initial}
\]

\[
\vec{r}(\frac{\pi}{2}) = \langle 0, e^{\frac{\pi}{2}} \rangle \quad \text{terminal}
\]

\[
\int_C \vec{F} \cdot d\vec{r} = f(0, e^{\frac{\pi}{2}}) - f(1, 0)
\]

\[
= \left( \frac{e^{\frac{3\pi}{2}}}{3} \right) - 4
\]

\[= \frac{e^{\frac{3\pi}{2}}}{3} - 4\]
Example 3.

\[ \mathbf{F} = (4 + 2xy, x^2 + y^2) \]
Example 4. Let \( \vec{F}(x, y) = (e^y + \cos y, x e^y - x \sin y) \).

(1) Determine whether or not \( \vec{F} \) is conservative.

\[
\begin{align*}
\frac{\partial P}{\partial y} &= e^y - \sin y \\
\Rightarrow \frac{\partial P}{\partial y} &= \frac{\partial Q}{\partial x} \\
\frac{\partial Q}{\partial x} &= e^y - \sin y
\end{align*}
\]

\( \Rightarrow \vec{F} \) is conservative.

(2) Find a function \( f \) such that \( \nabla f = \vec{F} \).

\[
\begin{align*}
f_x &= e^y + \cos y \\
f_y &= x e^y - x \sin y
\end{align*}
\]

\( \downarrow \)

\[
\begin{align*}
f_y &= x e^y - x \sin y + g'(y) \\
g'(y) &= 0 \\
\Rightarrow g(y) &= k
\end{align*}
\]

\( \Rightarrow f = xe^y + x \cos y + k \)

(3) Use part (2) to evaluate \( \int_C \vec{F} \cdot d\vec{r} \) where \( C \) is a curve from \((1, 0)\) to \((0, 3)\).

\[
\int_C \vec{F} \cdot d\vec{r} = f(0, 3) - f(1, 0) = 0 - (1 + 1) = -2
\]
**Example 5.** Let $\vec{F}(x, y) = (6x^2y + 2x \ln y)i + (2x^3 + \frac{x^2}{y})j$.

(1) Determine whether or not $\vec{F}$ is conservative.

\[ \frac{\partial P}{\partial y} = 6x^2 + \frac{2x}{y} \quad \Rightarrow \quad \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \quad \Rightarrow \quad \text{conservative} \]

\[ \frac{\partial Q}{\partial x} = 6x^2 + \frac{2x}{y} \]

(2) Find a function $f$ such that $\nabla f = \vec{F}$.

\[
\begin{align*}
    f_x &= 6x^2y + 2x \ln y \quad \Rightarrow \quad f(y) = 2x^3y + x^2 \ln y + g(y) \\
    f_y &= 2x^3 + \frac{x^2}{y} \\
\end{align*}
\]

\[ g'(y) = 0 \quad \Rightarrow \quad g(y) = k \]

So, $f(x, y) = 2x^3y + x^2 \ln y + k$

(3) Use part (2) to evaluate $\int_C \vec{F} \cdot dr$ where $C$ is a curve from $(1, 1)$ to $(0, 4)$.

\[ \int_C \vec{F} \cdot dr = f(0, 4) - f(1, 1) = 0 - (2 + 0) = -2 \]
Example 6. Let \( \vec{F}(x, y, z) = (yz, xz, xy + e^z) \).

(1) Find a function \( f \) such that \( \nabla f = \vec{F} \).

\[
\begin{align*}
\frac{\partial f}{\partial x} &= yz \\
\frac{\partial f}{\partial y} &= xz \\
\frac{\partial f}{\partial z} &= xy + e^z
\end{align*}
\]

\( \Rightarrow \text{So } f = xyz + e^z + K \)

\( f = xyz + g(y, z) \)

\[
\begin{align*}
\frac{\partial g}{\partial y} &= 0 \\
\frac{\partial g}{\partial z} &= e^z
\end{align*}
\]

\( \Rightarrow g(y, z) = e^z + K \)

(2) Use part (1) to evaluate \( \int_C \nabla f \cdot d\vec{r} \) where \( C \) is a line from \((1, 2, 0)\) to \((0, 3, 1)\)

\[
\int_C \nabla f \cdot d\vec{r} = f(0, 3, 1) - f(1, 2, 0)
\]

\[= e - (e^0)\]

\[= e - 1\]