1. Orthogonal Sets and Orthogonal Bases

Definition 1 (Orthogonal Set and Orthogonal Basis).

- A set \( \{\vec{u}_1, \ldots, \vec{u}_p\} \) of vectors in \( \mathbb{R}^n \) is called **orthogonal** if \( \vec{u}_i \cdot \vec{u}_j = 0 \) for any choice of indices \( i \neq j \).

- An **orthogonal basis** for a subspace \( W \) of \( \mathbb{R}^n \) is any basis for \( W \) which is also an orthogonal set.

Remark 2. An orthogonal set of nonzero vectors \( \{\vec{u}_1, \ldots, \vec{u}_p\} \) is always linearly independent.

Theorem 3 (Coordinates With Respect To An Orthogonal Basis).

Let \( B = \{\vec{u}_1, \ldots, \vec{u}_p\} \) be an orthogonal basis for a subspace \( W \) of \( \mathbb{R}^n \), and let \( \vec{y} \) be any vector in \( W \). Then

\[
\vec{y} = \left( \frac{\vec{y} \cdot \vec{u}_1}{\vec{u}_1 \cdot \vec{u}_1} \right) \vec{u}_1 + \cdots + \left( \frac{\vec{y} \cdot \vec{u}_p}{\vec{u}_p \cdot \vec{u}_p} \right) \vec{u}_p
\]

If \( W = \mathbb{R}^n \), then the \( B \)-coordinates of \( \vec{y} \) are given by:

\[
[\vec{y}]_B = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}
\]

with \( c_i = \frac{\vec{y} \cdot \vec{u}_i}{\vec{u}_i \cdot \vec{u}_i} = \frac{\vec{y} \cdot \vec{u}_i}{||\vec{u}_i||^2} \)

Example 4 (Orthogonal Sets and Bases).

\[
\vec{u}_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \quad \vec{u}_2 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}, \quad \vec{u}_3 = \begin{bmatrix} -1 \\ 7 \end{bmatrix}
\]

\[
\vec{u}_1 \cdot \vec{u}_2 = -3 + 2 + 1 = 0
\]

\[
\vec{u}_1 \cdot \vec{u}_3 = -3 - 4 + 7 = 0
\]

\[
\vec{u}_2 \cdot \vec{u}_3 = 1 - 8 + 7 = 0
\]

\( B = \{\vec{u}_1, \vec{u}_2, \vec{u}_3\} \) is an orthogonal basis for \( \mathbb{R}^2 \)

Find the coordinate of \( \vec{y} = \begin{bmatrix} 6 \\ -8 \end{bmatrix} \)

\[
\begin{align*}
C_1 &= \frac{\vec{y} \cdot \vec{u}_1}{\vec{u}_1 \cdot \vec{u}_1} = \frac{11}{11} = 1 \\
C_2 &= \frac{\vec{y} \cdot \vec{u}_2}{\vec{u}_2 \cdot \vec{u}_2} = \frac{-12}{6} = -2 \\
C_3 &= \frac{\vec{y} \cdot \vec{u}_3}{\vec{u}_3 \cdot \vec{u}_3} = \frac{-66}{66} = -1
\end{align*}
\]
2. Orthogonal Projection

- Given a nonzero vector $\vec{u}$ in $\mathbb{R}^n$, consider the problem of decomposing a vector $\vec{y}$ into a sum of two vectors

\[ \vec{y} = \vec{y}_1 + \vec{y}_2 \]

of which $\vec{y}_1 = c \cdot \vec{u}$ is a multiple of $\vec{u}$, and $\vec{y}_2$ is orthogonal to $\vec{u}$.

- Both $\vec{y}_1$ and $\vec{y}_2$ are easily found from their defining properties:

\[
\begin{align*}
\vec{y} \cdot \vec{u} &= (\vec{y}_1 + \vec{y}_2) \cdot \vec{u} \\
&= (c\vec{u}) \cdot \vec{u} + \vec{y}_2 \cdot \vec{u} \\
&= c(\vec{u} \cdot \vec{u}) + 0 \\
&= c\|\vec{u}\|^2.
\end{align*}
\]

- This shows that

\[
c = \frac{\vec{y} \cdot \vec{u}}{\vec{u} \cdot \vec{u}}, \quad \vec{y}_1 = \left(\frac{\vec{y} \cdot \vec{u}}{\vec{u} \cdot \vec{u}}\right) \vec{u}, \quad \vec{y}_2 = \vec{y} - \vec{y}_1 = \vec{y} - \left(\frac{\vec{y} \cdot \vec{u}}{\vec{u} \cdot \vec{u}}\right) \vec{u}.
\]

- If we replace $\vec{u}$ in the above calculation by a nonzero multiple of $\vec{u}$, say $d \cdot \vec{u}$, then the formulas for $\vec{y}_1$ and $\vec{y}_2$ remain unchanged, showing that they only depend on the line $\text{Span}\{\vec{u}\}$, rather than on $\vec{u}$ itself.
**Definition 5** (Orthogonal Projection Onto A Line). Let \( \vec{u} \) be a nonzero vector in \( \mathbb{R}^n \) and let \( L = \text{Span}\{ \vec{u} \} \) be the line in \( \mathbb{R}^n \) spanned by \( \vec{u} \). For a given vector \( \vec{y} \in \mathbb{R}^n \), the vectors

\[
\text{proj}_L(\vec{y}) = \left( \frac{\vec{y} \cdot \vec{u}}{\vec{u} \cdot \vec{u}} \right) \vec{u} \quad \text{and} \quad \vec{z} = \vec{y} - \text{proj}_L(\vec{y}),
\]

are called the **orthogonal projection of** \( \vec{y} \) **onto** \( L \) (or onto \( \vec{u} \)) and the **component of** \( \vec{y} \) **orthogonal to** \( L \) (or \( \vec{u} \)), respectively. For these two vectors one obtains

\[
\vec{y} = \text{proj}_L(\vec{y}) + \vec{z} \quad \text{and} \quad \vec{u} \cdot \vec{z} = 0.
\]

**Example 6** (Orthogonal Projections).

\[
\vec{y} = \begin{bmatrix} 7 \\ 6 \end{bmatrix}, \quad \vec{u} = \begin{bmatrix} 4 \\ 2 \end{bmatrix} \quad \text{Find the orthogonal projection of} \ \vec{y} \ \text{onto} \ \vec{u}.
\]

\[
\text{proj}_L(\vec{y}) = \frac{\vec{y} \cdot \vec{u}}{\vec{u} \cdot \vec{u}} \cdot \vec{u} = \frac{40}{20} \cdot \begin{bmatrix} 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 8 \\ 4 \end{bmatrix}
\]

\[
\vec{z} = \vec{y} - \text{proj}_L(\vec{y}) = \begin{bmatrix} 7 \\ 6 \end{bmatrix} - \begin{bmatrix} 8 \\ 4 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}
\]

\[
\vec{y} = \text{proj}_L(\vec{y}) + \vec{z}
\]
3. Orthonormal Sets and Bases

Definition 7 (Orthonormal Set & Basis, Orthogonal Matrix).

- An **orthonormal set** is an orthogonal set consisting of unit vectors.
- An **orthonormal basis** for a subspace $W$ of $\mathbb{R}^n$ is any basis for $W$ which is also an orthonormal set.
- An $m \times n$ matrix $U$ is called **orthogonal** if $U^T \cdot U = I_n$.

Remark 8. Every $n \times n$ orthogonal matrix $U$ is invertible with inverse matrix given by

$$U^{-1} = U^T$$

Moreover, $\det U = \pm 1$.

Theorem 9. The columns of an $m \times n$ matrix $U$ form an orthonormal set if and only if $U$ is orthogonal, that is if $U^T \cdot U = I_n$.

Theorem 10. Let $U$ be an $m \times n$ orthogonal matrix and let $\bar{x}$ and $\bar{y}$ be any vectors in $\mathbb{R}^n$. Then

1. $||U \cdot \bar{x}|| = ||\bar{x}||$.
2. $(U\bar{x}) \cdot (U\bar{y}) = \bar{x} \cdot \bar{y}$.
3. $(U\bar{x}) \cdot (U\bar{y}) = 0$ if and only if $\bar{x} \cdot \bar{y} = 0$.

Example 11 (Orthonormal Bases, Orthogonal Matrices).

\[
A = \frac{1}{3} \begin{bmatrix} 2 & -2 & 1 \\ 1 & 2 & 2 \\ -2 & 1 & -2 \end{bmatrix} \quad \quad A^T A = I_3
\]

\[
B = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \quad \quad B^T B = I_3
\]