Cohomology jump loci of configuration spaces

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Overview
Alexander Modules

- $X$: connected finite CW-complex.
- $G := \pi_1(X, x_0)$. 

The deck transformation group $G_{ab}$ acts on $X_{ab}$.

The "Crowell exact sequence" of $X$ as $\mathbb{Z}[G_{ab}]$-modules:

$$0 \rightarrow H_1(X_{ab}; \mathbb{Z}) \rightarrow H_1(X_{ab}, F; \mathbb{Z}) \rightarrow I(G_{ab}) \rightarrow 0$$

where $I(G_{ab}) = \ker \epsilon: \mathbb{Z}[G_{ab}] \rightarrow \mathbb{Z}$.

Alexander module $A(G) := H_1(X_{ab}, F; \mathbb{Z})$.

Alexander invariant $B(G) = H_1(X_{ab}; \mathbb{Z}) = G'/G''$, where $G'' = [G', G']$ is the second derived subgroup.

The $\mathbb{Z}[G_{ab}]$-module structure on $B(G)$ is determined by the extension:

$$0 \rightarrow G'/G'' \rightarrow G/G'' \rightarrow G/G' \rightarrow 0$$

with $G/G'$ acting on the cosets of $G''$ via conjugation:

$$gG' \cdot hG'' = ghg^{-1}G''$$

for $g \in G$, $h \in G'$. 

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Chen Lie algebra

- The lower central series $G$: $\Gamma_1 G = G$, $\Gamma_{k+1} G = [\Gamma_k G, G]$, $k \geq 1$.
- The Chen Lie algebra of a group $G$ is defined to be
  \[
  \text{gr}(G / G''; k) := \bigoplus_{k \geq 1} (\Gamma_k (G / G'') / \Gamma_{k+1} (G / G'')) \otimes_{\mathbb{Z}} k.
  \]
- The quotient map $h: G \to G / G''$ induces $\text{gr}(G; k) \to \text{gr}(G / G''; k)$.
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- \( \theta_k (F_n) = (k - 1) \binom{n + k - 2}{k} \), \( k \geq 2 \). [Chen51]
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- The module $B(G)$ has an $I$-adic filtration $\{I^k B(G)\}_{k \geq 0}$.
- $\text{gr}(B(G)) = \bigoplus_{k \geq 0} I^k B(G)/I^{k+1} B(G)$ is a graded $\text{gr}(\mathbb{Z}[G_{\text{ab}}])$-module.
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**Proposition (Massey 80)**

For each $k \geq 2$, there exists an isomorphism
\[ \text{gr}_k (G/G'') \cong \text{gr}_{k-2} (B(G)). \]
Alexander varieties

Definition (Libgober 1992)

The *Alexander variety* of $X$ (over $\mathbb{C}$)

$$W^i_d(X,\mathbb{C}) = V(E_{d-1}(H_i(X^{ab},\mathbb{C})))$$

is the subvariety of $\mathbb{T}(X)$, defined by the Fitting ideals.
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- $\mathcal{W}_1^1(T^n, \mathbb{C}) = \{1\}.$
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- $\mathcal{W}_d^1(\Sigma_g, \mathbb{C}) = (\mathbb{C}^*)^{2g}$ for $g > 1$, $d < 2g - 1$. 
Example (Borromean rings)

Let $X$ be the complement in $S^3$ of the Borromean rings: 

A presentation for the fundamental group

$$G = \pi_1(X) = \langle x, y, z \mid zyz^{-1}xzy^{-1}z^{-1} = yxy^{-1}, xzx^{-1}yxz^{-1}x = zyz^{-1} \rangle.$$
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- $\mathbb{C}[G_{ab}] = \mathbb{C}[t_1^{\pm 1}, t_2^{\pm 1}, t_3^{\pm 1}]$.
- $A(G) = \text{coker} \begin{pmatrix}
0 & (t_3 - 1)(1 - t_1) & (1 - t_1)(1 - t_2) \\
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- $B(G) = \text{coker } \begin{pmatrix} 0 & t_2 - 1 & 0 \\ 0 & 0 & t_1 - 1 \end{pmatrix}$. 

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- $B(G) = \text{coker} \begin{pmatrix} t_3 - 1 & 0 & 0 \\ 0 & t_2 - 1 & 0 \\ 0 & 0 & t_1 - 1 \end{pmatrix}$
- The Alexander variety
  $$\mathcal{W}_1^1(X, \mathbb{C}) = \{ t_1 = 1 \} \cup \{ t_2 = 1 \} \cup \{ t_3 = 1 \} = (\mathbb{C}^*)^2 \cup (\mathbb{C}^*)^2 \cup (\mathbb{C}^*)^2;$$
  $$\mathcal{W}_2^1(X, \mathbb{C}) = \{ t_1 = t_2 = 1 \} \cup \{ t_2 = t_3 = 1 \} \cup \{ t_3 = t_1 = 1 \};$$
  $$\mathcal{W}_3^1(X, \mathbb{C}) = \{ 1 \}.$$
The characteristic varieties

- The **rank 1 local system** on $X$ is a 1-dimensional $\mathbb{C}$-vector space $\mathbb{C}_\rho$ with a right $\mathbb{C}G$-module structure $\mathbb{C}_\rho \times G \to \mathbb{C}_\rho$ given by $\rho(g) \cdot a$ for $a \in \mathbb{C}_\rho$ and $g \in G$ for $\rho \in \text{Hom}(G, \mathbb{C}^*)$. 

$H_i(X, \mathbb{C}_\rho) := H_i(\mathbb{C}^*(\tilde{X}, \mathbb{C}) \otimes \mathbb{C}_G \mathbb{C}_\rho)$ the homology group of $X$ with coefficient $\mathbb{C}_\rho$.

**Definition**

The **characteristic varieties** of $X$ over $\mathbb{C}$ are the Zariski closed subsets $V_i^{C}(X) = \{ \rho \in T(X) = \text{Hom}(G, \mathbb{C}^*) | \dim \mathbb{C}_\rho \geq d \}$ for $i \geq 1$ and $d \geq 1$.

**Proposition (Papadima, Suciu10)**

$q \bigcup_{i=0}^{\infty} V_i^{1}(X) = q \bigcup_{i=0}^{\infty} W_i^{1}(X)$. 

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The characteristic varieties of $X$ over $\mathbb{C}$ are the Zariski closed subsets

$$\mathcal{V}_d^i(X, \mathbb{C}) = \{ \rho \in \mathbb{T}(X) = \text{Hom}(G, \mathbb{C}^*) \mid \dim_{\mathbb{C}} H_i(X, \mathbb{C}_\rho) \geq d \}$$

for $i \geq 1$ and $d \geq 1$. 
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**Definition**

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$$V^i_d(X, \mathbb{C}) = \{ \rho \in T(X) = \text{Hom}(G, \mathbb{C}^*) \mid \dim_{\mathbb{C}} H_i(X, \mathbb{C}_\rho) \geq d \}$$

for $i \geq 1$ and $d \geq 1$.

**Proposition (Papadima,Suciu10)**

$$\bigcup_{i=0}^{q} V^i_1(X, \mathbb{C}) = \bigcup_{i=0}^{q} \mathcal{W}^i_1(X, \mathbb{C}).$$
The resonance varieties

- $A = H^*(G, \mathbb{C})$. For each $a \in A^1$, we have $a^2 = 0$. 
The resonance varieties

- $A = H^* (G, \mathbb{C})$. For each $a \in A^1$, we have $a^2 = 0$.
- Define a cochain complex of finite-dimensional $\mathbb{C}$-vector spaces,

$$(A, a) : A^0 \xrightarrow{a \cup -} A^1 \xrightarrow{a \cup -} A^2 \xrightarrow{a \cup -} \cdots ,$$

with differentials given by left-multiplication by $a$. 

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**Definition**

The *resonance varieties* of $G$ are the homogeneous subvarieties of $A^1$

$$ \mathcal{R}^i_d(G, \mathbb{C}) = \{ a \in A^1 | \dim_{\mathbb{C}} H^i(A; a) \geq d \}, $$

defined for all integers $i \geq 1$ and $d \geq 1$. 
The resonance varieties

- \( A = H^*(G, \mathbb{C}) \). For each \( a \in A^1 \), we have \( a^2 = 0 \).
- Define a cochain complex of finite-dimensional \( \mathbb{C} \)-vector spaces,

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defined for all integers \( i \geq 1 \) and \( d \geq 1 \).

- \( \mathcal{R}^1_1(T^n, \mathbb{C}) = \{0\} \);
- \( \mathcal{R}^1_1(\Sigma_g, \mathbb{C}) = \mathbb{C}^{2g}, \ g \geq 2 \).
1-Formality and Tangent Cone Theorem

- A space $X$ is 1-formal if there exists a cdga morphism from the minimal model $\mathcal{M}(X)$ to $(H^*(X, \mathbb{Q}), 0)$ inducing isomorphism in cohomology of degree 1 and monomorphism in degree 2.

Example (Borromean link again)

$$R_1 d(G, C) = H_1(X; C) = C^3$$ for $d \leq 3$. 

$$TC_1(V_1 d(G, C)) = \{x_1 = 0\} \cup \{x_2 = 0\} \cup \{x_3 = 0\}.$$ 

$\Rightarrow X$ is not 1-formal.
1-Formality and Tangent Cone Theorem

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- A group $G$ is **1-formal** if the Eilenberg-MacLane space $K(G, 1)$ is 1-formal.

Example (Borromean link again): $R^1d(G, C) = H^1(G, C) = C^3$ for $d \leq 3$. 

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1-Formality and Tangent Cone Theorem

- A space $X$ is $1$-formal if there exists a cdga morphism from the minimal model $\mathcal{M}(X)$ to $(H^*(X, \mathbb{Q}), 0)$ inducing isomorphism in cohomology of degree 1 and monomorphism in degree 2.
- A group $G$ is $1$-formal if the Eilenberg-MacLane space $K(G, 1)$ is 1-formal.

**Theorem (Dimca, Papadima, Suciu 09)**

If $G$ is 1-formal, then the tangent cone $TC_1(V^1_d(G, \mathbb{C}))$ equals $R^1_d(G, \mathbb{C})$. Moreover, $R^1_d(G, \mathbb{C})$ is a union of rationally defined linear subspaces of $H^1(G, \mathbb{C})$. 

⇒ $X$ is not 1-formal.
1-Formality and Tangent Cone Theorem

- A space $X$ is 1-formal if there exists a cdga morphism from the minimal model $\mathcal{M}(X)$ to $(H^*(X, \mathbb{Q}), 0)$ inducing isomorphism in cohomology of degree 1 and monomorphism in degree 2.
- A group $G$ is 1-formal if the Eilenberg-MacLane space $K(G, 1)$ is 1-formal.

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Example (Borromean link again)

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**Theorem (Dimca, Papadima, Suciu 09)**

*If $G$ is 1-formal, then the tangent cone $\text{TC}_1(\mathcal{V}_d^1(G, \mathbb{C}))$ equals $\mathcal{R}_d^1(G, \mathbb{C})$. Moreover, $\mathcal{R}_d^1(G, \mathbb{C})$ is a union of rationally defined linear subspaces of $H^1(G, \mathbb{C})$.***

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  \[ \Rightarrow X \text{ is not 1-formal.} \]
The configuration spaces
Let $M$ be a connected manifold with $\dim_{\mathbb{R}} M \geq 2$. The configuration space

$$\mathcal{F}(M, n) = \{(x_1, \cdots, x_n) \in M \times \cdots \times M \mid x_i \neq x_j \text{ for } i \neq j\}.$$  

There is a free action of $S_n$ on $\mathcal{F}(M, n)$ by permutation of coordinates, with orbit space $\mathcal{C}(M, n) = \mathcal{F}(M, n)/S_n$. 

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- Example: The braid group $B_n = \pi_1(C(\mathbb{R}^2, n))$ and pure braid group $P_n = \pi_1(\mathcal{F}(\mathbb{R}^2, n))$ with $1 \to P_n \to B_n \xrightarrow{\rho} S_n \to 1$. 

Proposition (Cohen, Suciu 95)

The Chen ranks of $P_n$ are given by

$$ \theta_1(P_n) = \binom{n}{2}; \quad \theta_2(P_n) = \binom{n}{3}; \quad \theta_k(P_n) = \binom{k-1}{n+1}, $$

for $k \geq 3$.
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**Corollary**

$P_n$ is not isomorphic to $\Pi_n = F_1 \times \cdots \times F_{n-1}$ for $n \geq 4$. 

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The pure braid groups on Riemann surface

- \( P_{g,n} = \pi_1(\mathcal{F}(\Sigma_g, n)) \), where \( \mathcal{F}(\Sigma_g, n) \) is the configuration of \( \Sigma_g \), which is a smooth compact complex curve of genus \( g \) (\( g \geq 1 \)).
The pure braid groups on Riemann surface

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**Proposition (Dimca, Papadima, Suciu 09)**

The (first) resonance variety of $P_{1,n}$ is

$$\mathcal{R}_1^1(P_{1,n}, \mathbb{C}) = \left\{ (x, y) \in \mathbb{C}^n \times \mathbb{C}^n \mid \sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i = 0, x_i y_j - x_j y_i = 0, \text{ for } 1 < i < j \leq n \right\}$$
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\sum_{i=1}^{n} x_i &= \sum_{i=1}^{n} y_i = 0 \\
 x_i y_j - x_j y_i &= 0, \text{ for } 1 < i < j \leq n
\end{align*} \right\}
\]

**Corollary**

\( P_{n,1} \) is not 1-formal for \( n \geq 3 \).
The pure virtual braid groups

- The virtual braids come from the virtual knot theory by Kauffman.

The generators $\sigma_i$ and $s_i$ of the virtual braid groups $vB_n$ are

$$1_i - 1_i + 1_i + 2 \cdots \cdots \circ 1_i - 1_i + 1_i + 2 \cdots \cdots$$

The relations for $vB_n$ include the relations for $B_n$ and $S_n$, and

$$\{ \sigma_i s_j = s_j \sigma_i, \mid i - j \mid \geq 2 \}, \ s_i s_{i+1} \sigma_i = \sigma_i + 1 s_i s_{i+1}, \ i = 1, \ldots, n-2.$$ (1)

$1 \rightarrow vP_n \rightarrow vB_n \rho \rightarrow S_n \rightarrow 1.$

The pure virtual braid groups $vP_n$ has presentation [Bardakov04]

$$\langle x_{ij}, (1 \leq i \neq j \leq n) \bigg| \left| x_{ij} x_{ik} x_{jk} = x_{jk} x_{ik} x_{ij} ; x_{ij} x_{kl} = x_{kl} x_{ij} ; i, j, k, l \text{ distinct} \right. \rangle.$$ $vP_n^+$ is the quotient of $vP_n$ by the relations $x_{ij} x_{ji} = 1$ for $i \neq j$. He Wang (Joint with Alexander Suciu)

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\[
\begin{array}{cccccc}
1 & i-1 & i & i+1 & i+2 & n \\
\cdots & \x & \cdots & \cdots & \cdots & \cdots \\
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\cdots & \x & \cdots & \cdots & \cdots & \cdots \\
\end{array}
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\vdots & \boxtimes & \vdots & \vdots & \vdots & \\
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- The relations for $vB_n$ include the relations for $B_n$ and $S_n$, and

$$
\begin{aligned}
\sigma_i s_j &= s_j \sigma_i, & |i - j| \geq 2, \\
s_i s_{i+1} \sigma_i &= \sigma_{i+1} s_i s_{i+1}, & i = 1, \ldots, n - 2.
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\ldots & \times & \times & \ldots & & \\
\end{array}
\quad
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1 & i-1 & i & i+1 & i+2 & n \\
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(1)

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- The pure virtual braid groups $vP_n$ has presentation [Bardakov04]

\[
\left\langle x_{ij}, (1 \leq i \neq j \leq n) \right| \begin{array}{l}
x_{ij} x_{ik} x_{jk} = x_{jk} x_{ik} x_{ij}; \\
x_{ij} x_{kl} = x_{kl} x_{ij}; \quad i, j, k, l \text{ distinct}
\end{array} \right\}.
\]
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\[
\begin{array}{c|c|c|c|c|c|c}
1 & i-1 & i & i+1 & i+2 & n \\
\vdots & \cdots & \times & \cdots & \cdots & \cdots \\
\end{array}
\quad
\begin{array}{c|c|c|c|c|c|c}
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\vdots & \cdots & \times & \cdots & \cdots & \cdots \\
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\]

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\sigma_is_j = s_j\sigma_i, & |i-j| \geq 2, \\
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\]

- $vP_n^+$ is the quotient of $vP_n$ by the relations $x_{ij}x_{ji} = 1$ for $i \neq j$. 

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Theorem (Suciu, W. 15)

The pure virtual braid groups $vP_n$ and $vP^+_n$ are 1-formal if and only if $n \leq 3$. 

Sketch of proof: 

Lemma 

There are split monomorphisms 

$\begin{array}{c} vP_2 \\ \downarrow \\ vP_3 \\ \downarrow \\ vP_4 \\ \downarrow \\ \vdots \\ vP_5 \\ \downarrow \\ vP_6 \\ \downarrow \\ \vdots \end{array}$

Lemma 

Suppose there is a split monomorphism $\iota: N \hookrightarrow G$. If $G$ is 1-formal, then $N$ is also 1-formal.

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$$
\begin{align*}
\text{vP}_2^+ & \xleftarrow{} \text{vP}_3^+ & \xleftarrow{} \text{vP}_4^+ & \xleftarrow{} \text{vP}_5^+ & \xleftarrow{} \text{vP}_6^+ & \rightarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \\
\text{vP}_2 & \xleftarrow{} \text{vP}_3 & \xleftarrow{} \text{vP}_4 & \xleftarrow{} \text{vP}_5 & \xleftarrow{} \text{vP}_6 & \rightarrow & \cdots 
\end{align*}
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Suppose there is a split monomorphism $\iota: N \hookrightarrow G$. If $G$ is 1-formal, then $N$ is also 1-formal.
Lemma

The group $\mu P_3$ is 1-formal.

Next we show that $\mu P_3 + 4$ is not 1-formal.
Lemma

The group $vP_3$ is 1-formal.

Next we show that $vP_4^+$ is not 1-formal.
Lemma

The group $vP_3$ is 1-formal.

Next we show that $vP_4^+$ is not 1-formal.

Lemma

The first resonance variety $\mathcal{R}_1^1(vP_4^+, \mathbb{C})$ is the subvariety of $\mathbb{C}^6$ given by the equations

\[
\begin{align*}
    x_{12}x_{24}(x_{13} + x_{23}) + x_{13}x_{34}(x_{12} - x_{23}) - x_{24}x_{34}(x_{12} + x_{13}) &= 0, \\
    x_{12}x_{23}(x_{14} + x_{24}) + x_{12}x_{34}(x_{23} - x_{14}) + x_{14}x_{34}(x_{23} + x_{24}) &= 0, \\
    x_{13}x_{23}(x_{14} + x_{24}) + x_{14}x_{24}(x_{13} + x_{23}) + x_{34}(x_{13}x_{23} - x_{14}x_{24}) &= 0, \\
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\]
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\quad x_{13}x_{23}(x_{14} + x_{24}) + x_{14}x_{24}(x_{13} + x_{23}) + x_{34}(x_{13}x_{23} - x_{14}x_{24}) &= 0, \\
\quad x_{12}(x_{13}x_{14} - x_{23}x_{24}) + x_{34}(x_{13}x_{23} - x_{14}x_{24}) &= 0.
\end{align*}
\]

$\Rightarrow$ The group $vP_4^+$ is not 1-formal.
The pure welded braid groups (McCool groups)

- The welded braid group $wB_n$ has the same generators as $vB_n$, adding one more class of relations

$$\sigma_i \sigma_{i+1} s_i = s_{i+1} \sigma_i \sigma_{i+1}, \ i = 1, 2, \ldots, n - 2.$$ 

- $1 \to wP_n \to wB_n \xrightarrow{\rho} S_n \to 1$.

- The pure welded braid groups $wP_n$ has presentation [McCool 86]

$$\left\langle x_{ij}, (1 \leq i \neq j \leq n) \mid x_{ij} x_{ik} x_{jk} = x_{jk} x_{ik} x_{ij}; \ x_{ij} x_{kl} = x_{kl} x_{ij}; \ i, j, k, l \text{ distinct} \right. \left. \ x_{ij} x_{kj} = x_{kj} x_{ij}; \ i, j, k \text{ distinct} \right\rangle.$$ 

- There is a subgroup of $wP_n$ generated by the $x_{ij}$ for $1 \leq i < j \leq n$, denoted by $wP_n^+$. The group $wP_n$ is called McCool group and $wP_n^+$ is called upper McCool group.
Theorem (D. Cohen 09)

The first resonance variety of McCool group $\text{wP}_n$ is

$$R_1^1(\text{wP}_n, \mathbb{C}) = \bigcup_{1 \leq i < j \leq n} C_{ij} \cup \bigcup_{1 \leq i < j < k \leq n} C_{ijk},$$

where $C_{ij} = \mathbb{C}^2$ and $C_{ijk} = \mathbb{C}^3$. 

Theorem (Suciu, W. 15)

The first resonance variety of upper McCool group $\text{wP}_n^+$ is

$$R_1^1(\text{wP}_n^+, \mathbb{C}) = \bigcup_{1 \leq i < j \leq n-1} C_{ij},$$

where $C_{ij} = \mathbb{C}^{j+1}$. 

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Theorem (D. Cohen 09)

The first resonance variety of McCool group $wP_n$ is

$$R_1^1(wP_n, \mathbb{C}) = \bigcup_{1 \leq i < j \leq n} C_{ij} \cup \bigcup_{1 \leq i < j < k \leq n} C_{ijk},$$

where $C_{ij} = \mathbb{C}^2$ and $C_{ijk} = \mathbb{C}^3$.

Theorem (Suciu, W. 15)

The first resonance variety of upper McCool group $wP_n^+$ is

$$R_1^1(wP_n^+, \mathbb{C}) = \bigcup_{1 \leq i < j \leq n-1} C_{i,j},$$

where $C_{i,j} = \mathbb{C}^{j+1}$. 
Future work

- The relations between the Chen ranks $\theta_k(G)$ and $R_1^1(G)$

$$\theta_k(G) = \sum_{m \geq 2} c_m \cdot \theta_k(F_m)$$

where $c_m$ is the number of $m$-dimensional components of $R_1^1(G)$. (Schenck and Suciu04) (Cohen and Schenck14)
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- The relations between the Chen ranks $\theta_k(G)$ and $V_1^1(G)$. Replace $c_m$ in the above formula by the number of $m$-dimensional components of $TC_1(V_1(G))$. 
Future work

- The relations between the Chen ranks $\theta_k(G)$ and $R_1(G)$

$$\theta_k(G) = \sum_{m \geq 2} c_m \cdot \theta_k(F_m)$$

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- The relations between the Chen ranks $\theta_k(G)$ and $\mathcal{V}_1(G)$. Replace $c_m$ in the above formula by the number of $m$-dimensional components of $TC_1(\mathcal{V}_1(G))$.

Thank You!