Formality properties: generalizations and applications

He Wang
(joint work with Alex Suciu)

University of Nevada, Reno

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Rational homotopy theory

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- Two spaces $X$ and $Y$ have the same rational homotopy type if there is a continuous map $f: X \to Y$ inducing an isomorphism

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$$\pi_\ast(f) \otimes \mathbb{Q} : \pi_\ast(X) \otimes \mathbb{Q} \to \pi_\ast(Y) \otimes \mathbb{Q}.$$ 

- For a “formal” simply connected space, its rational homotopy type is determined by its cohomology algebra over $\mathbb{Q}$. 

Formality property of a CDGA

- Let $A = (A^*, d_A)$ be a graded-commutative differential graded algebra (CDGA) over $\mathbb{Q}$. 

- A morphism $f: A \rightarrow B$ is a quasi-isomorphism if $f^*: H^*(A) \rightarrow H^*(B)$ is an isomorphism.

- Each connected CDGA $(A, d_A)$ has a minimal model $(M(A), d)$, unique up to isomorphism. [Sullivan 77]

- $A$ is said to be formal if there exists a quasi-isomorphism $(M(A), d) \rightarrow (H^*(A), 0)$, equivalently, there is a sequence of zig-zag quasi-isomorphisms $(A, d_A) \leftarrow \cdots \leftarrow (H^*(A), 0)$.
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Theorem (Deligne–Griffiths–Morgan–Sullivan 75) Compact K"ahler manifolds are formal over $\mathbb{R}$.

Theorem (Sullivan 77, Neisendorfer–Miller 78, Halperin–Stasheff 79) Let $\mathbb{Q} \subset K$ be a field extension, and $X$ be a connected space with finite Betti numbers. $X$ is formal over $\mathbb{Q}$ if and only if $X$ is formal over $K$.

Corollary [Sullivan 77] Compact K"ahler manifolds are formal over $\mathbb{Q}$. 
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Formality Properties of DG-

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Theorem (Santos–Navarro–Pascual–Roig 05)

Let $\mathbb{Q} \subset \mathbb{K}$ be a field extension, and $P$ be a dg operad over $\mathbb{Q}$ with homology of finite type. $P$ is formal if and only if $P \otimes \mathbb{K}$ is formal.
Partial formality

- A CDGA morphism \( f : A \to B \) is an \( i \)-quasi-isomorphism if

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f^* : H^j(A) \to H^j(B)
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- Each connected **CDGA** $A$ has an $i$-minimal model $\mathcal{M}(A, i)$ unique up to isomorphism. [Morgan 78]
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Theorem (Suciu–W.)

Let $\mathbb{Q} \subset \mathbb{K}$ be a field extension, and $X$ be a connected space with finite Betti numbers $b_1(X), \ldots, b_{i+1}(X)$. Then $X$ is $i$-formal over $\mathbb{Q}$ if and only if $X$ is $i$-formal over $\mathbb{K}$. 
1-formality of groups

The 1-formality of a path-connected space $X$ depends only on $\pi_1(X)$. A finitely generated group $G$ is called 1-formal if $X = K(G,1)$ is 1-formal, i.e., $\mathcal{M}(X,1)$ is 1-quasi-isomorphic to $(H^\ast(G;\mathbb{Q}),0)$.

Example

Formal spaces: compact Kähler manifolds, complements of complex hyperplane arrangements, ...

1-formal groups: finitely generated Artin groups, pure braid groups, ...

Possible not 1-formal groups: link groups, nilpotent groups, pure braid groups on surfaces, the fundamental groups of algebraic varieties, ...

Heisenberg (type) group $H_n$ is $(n-1)$-formal but not $n$-formal.

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Graded Lie algebras

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- The *associated graded Lie algebra* of $G$ is defined to be

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  \mathfrak{h}(G; \mathbb{Q}) := \text{Lie}(H_1(G; \mathbb{Q}))/\langle \text{im}(\partial_G) \rangle.
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  Here, $\partial_G$ is the dual of $H^1(G; \mathbb{Q}) \wedge H^1(G; \mathbb{Q}) \cup H^2(G; \mathbb{Q})$. There exists an epimorphism $\Phi_G : \mathfrak{h}(G; \mathbb{Q}) \twoheadrightarrow \text{gr}(G; \mathbb{Q})$. We say that a group $G$ is *graded-formal*, if $\Phi_G : \mathfrak{h}(G; \mathbb{Q}) \twoheadrightarrow \text{gr}(G; \mathbb{Q})$ is an isomorphism of graded Lie algebras.
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Malcev Lie algebra

Let $G$ be a finitely generated group.

- There exists a tower of nilpotent Lie algebras [Malcev 51]

\[
\mathcal{L}((G/\Gamma_2 G) \otimes \mathbb{Q}) \leftarrow \mathcal{L}((G/\Gamma_3 G) \otimes \mathbb{Q}) \leftarrow \mathcal{L}((G/\Gamma_4 G) \otimes \mathbb{Q}) \leftarrow \]

The inverse limit of the tower is called the Malcev Lie algebra of $G$, denoted by $\mathfrak{m}(G; \mathbb{Q})$.

The universal enveloping algebra of $\mathfrak{m}(G; \mathbb{Q})$ is isomorphic to $\hat{\mathbb{Q}} G$.

[Quillen 69]

Let $M(G, 1)$ be the 1-minimal model of $K(G, 1)$.

There is a one-to-one correspondence between $M(G, 1)$ and the Malcev Lie algebra $\mathfrak{m}(G; \mathbb{Q})$. [Sullivan 77, Cenkl–Porter 81]
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- Let $\mathcal{M}(G, 1)$ be the 1-minimal model of $K(G, 1)$. These is a one to one corresponding between $\mathcal{M}(G, 1)$ and the Malcev Lie algebra $m(G; \mathbb{Q})$. [Sullivan 77, Cenkl–Porter 81]
Partial formality of groups

- $\text{gr}(G; \mathbb{Q}) \xrightarrow{\cong} \text{gr}(m(G; \mathbb{Q}))$. [Quillen 68]
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- $\text{gr}(G; \mathbb{Q}) \xrightarrow{\cong} \text{gr}(m(G; \mathbb{Q}))$. [Quillen 68]
- A group $G$ is 1-formal iff $m(G; \mathbb{Q}) \cong \hat{h}(G; \mathbb{Q})$. [Markl–Papadima 92]
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- $\text{gr}(G; \mathbb{Q}) \xrightarrow{\cong} \text{gr}(\mathfrak{m}(G; \mathbb{Q}))$. [Quillen 68]
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- We say that a group $G$ is \textit{filtered-formal}, if there is a filtered Lie algebra isomorphism $\mathfrak{m}(G; \mathbb{Q}) \cong \mathring{\text{gr}}(G; \mathbb{Q})$. 
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\begin{align*}
\text{m}(G; \mathbb{Q}) & \xrightarrow{1\text{-formal}} \hat{\mathfrak{h}}(G; \mathbb{Q}) \\
\hat{\text{gr}}(\text{m}(G; \mathbb{Q})) & \xrightarrow{\cong \text{Quillen}} \hat{\text{gr}}(G; \mathbb{Q}).
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m(G; \mathbb{Q}) & \xrightarrow{1\text{-formal}} & \hat{h}(G; \mathbb{Q}) \\
\text{filtered-formal} & \downarrow & \text{graded-formal} \\
\hat{\text{gr}}(m(G; \mathbb{Q})) & \xrightarrow{\cong} & \hat{\text{gr}}(G; \mathbb{Q}).
\end{array}
\]

- formal \( \implies \) \( i \)-formal \( \implies \) 1-formal \( \iff \) graded-formal + filtered-formal.
A finitely generated group $G$ is filtered-formal (graded-formal) over $\mathbb{Q}$ if and only if it is filtered-formal (graded-formal) over $K$.

Remark

The filtered formality of finite-dimensional, nilpotent Lie algebras has been studied under many different names: 'Carnot', 'naturally graded', 'homogeneous' and 'quasi-cyclic'. In this special case, the above theorem was proved by Cornulier (14).

A finitely generated, torsion-free, 2-step nilpotent group is filtered-formal. [Suciu-W.]

Recently, Bar-Natan has explored the "Taylor expansion" of $G$, which is a map $E: G \to \hat{\text{gr}}(\mathbb{Q}G)$ satisfying some properties. $G$ has a Taylor expansion $\iff G$ is filtered-formal. $G$ has a quadratic Taylor expansion $\iff G$ is 1-formal.

A map $T: F_n \to \mathbb{Q}\langle\langle z_1, \cdots, z_n \rangle\rangle$ defined by $T(x_i) = \exp(z_i)$ is a Taylor expansion of $F_n$. 
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Proposition (Suciu–W.)

Let $G$ be a finitely generated group, and let $K \leq G$ be a finitely generated subgroup. Suppose there is a split monomorphism $\iota : K \rightarrow G$. Then:

1. If $G$ is graded-formal, then $K$ is also graded-formal.
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3. If $G$ is $1$-formal, then $K$ is also $1$-formal.

Proposition (Suciu–W.)

Let $G_1$ and $G_2$ be two finitely generated groups. The following conditions are equivalent.

1. $G_1$ and $G_2$ are graded-formal (respectively, filtered-formal, or $1$-formal).
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Propagating partial formality properties

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An obstruction to 1-formality: resonance varieties

- Suppose $A^* := H^*(G, \mathbb{C})$ has finite dimension in each degree.
An obstruction to 1-formality: resonance varieties

- Suppose $A^* := H^*(G, \mathbb{C})$ has finite dimension in each degree.
- For each $a \in A^1$, define a cochain complex of finite-dimensional $\mathbb{C}$-vector spaces,

$$(A, a) : A^0 \xrightarrow{a} A^1 \xrightarrow{a} A^2 \xrightarrow{a} \cdots,$$

with differentials given by left-multiplication by $a$. 

The resonance varieties of $G$ are the homogeneous subvarieties of $A^1$.
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$$ \mathcal{R}_k^i(G, \mathbb{C}) = \{ a \in A^1 \mid \dim_{\mathbb{C}} H^i(A^*; a) \geq k \}. $$

**Theorem (Dimca–Papadima–Suciu 09)**

*If $G$ is 1-formal, $\mathcal{R}_k^1(G, \mathbb{C})$ is a union of rationally defined linear subspaces of $H^1(G, \mathbb{C})$.***
Pure virtual braid groups

- The virtual braid groups come from the virtual knot theory introduced by Kauffman(99).
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The *pure virtual braid groups* of $vP_n$ has a presentation [Bardakov (04)] with generators $x_{ij}$ for $1 \leq i \neq j \leq n$, subject to the relations

$$x_{ij} x_{ik} x_{jk} = x_{jk} x_{ik} x_{ij},$$

$$[x_{ij}, x_{st}] = 1,$$

for $i, j, k$ distinct, for $i, j, s, t$ distinct.
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$vP_n$ has a subgroup generated by $x_{ij}$ for $i < j$, denoted by $vP_n^+$. 
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Bartholdi, Enriquez, Etingof, and Rains (06) independently studied $vP_n$ and $vP_n^+$ as groups arising from the Yang-Baxter equations.

They also showed that $vP_n$ and $vP_n^+$ are graded-formal (with the work of P. Lee (13)) and computed the cohomology algebras of these groups.
Theorem (Suciu–W.)

The pure virtual braid groups $vP_n$ and $vP_n^+$ are 1-formal if and only if $n \leq 3$. 
Theorem (Suciu–W.)

The pure virtual braid groups \( vP_n \) and \( vP_n^+ \) are 1-formal if and only if \( n \leq 3 \).

Sketch of proof:

Lemma

There are split monomorphisms

\[
\begin{array}{cccccc}
vP_2^+ & \longrightarrow & vP_3^+ & \longrightarrow & vP_4^+ & \longrightarrow & vP_5^+ & \longrightarrow & vP_6^+ & \longrightarrow & \ldots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
vP_2 & \longrightarrow & vP_3 & \longrightarrow & vP_4 & \longrightarrow & vP_5 & \longrightarrow & vP_6 & \longrightarrow & \ldots 
\end{array}
\]
Lemma

The group $vP_3$ is 1-formal.

Proof:

$P_3 \cong N \times Z$.

□

Lemma

The group $vP_4$ is not 1-formal.

Proof: The first resonance variety $R_1(vP_4, C)$ is the variety of $C_6$ given by the equations:

$$x_2 x_4 (x_3 + x_4) + x_3 x_4 (x_2 + x_3) - x_2 x_3 (x_1 + x_2) = 0,$$

$$x_1 x_3 (x_1 + x_3) + x_2 x_3 (x_2 + x_3) + x_2 x_4 (x_2 + x_3) = 0,$$

$$x_1 x_2 (x_1 + x_2) - x_2 x_3 (x_4 - x_3) = 0.$$

$\Rightarrow$ The group $vP_4$ is not 1-formal.

□
Lemma

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Proof: $vP_3 \cong N \ast \mathbb{Z}$, and $P_4 \cong N \times \mathbb{Z}$. □
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The group $vP_3$ is $1$-formal.

Proof: $vP_3 \cong N \ast \mathbb{Z}$, and $P_4 \cong N \times \mathbb{Z}$. \qed

Lemma

The group $vP_4^+$ is not $1$-formal.
Lemma

The group $vP_3$ is 1-formal.

Proof: $vP_3 \cong N \times \mathbb{Z}$, and $P_4 \cong N \times \mathbb{Z}$.

Lemma

The group $vP_4^+$ is not 1-formal.

Proof: The first resonance variety $R_1(vP_4^+, \mathbb{C})$ is the subvariety of $\mathbb{C}^6$ given by the equations

\begin{align*}
    x_{12}x_{24}(x_{13} + x_{23}) + x_{13}x_{34}(x_{12} - x_{23}) - x_{24}x_{34}(x_{12} + x_{13}) &= 0, \\
    x_{12}x_{23}(x_{14} + x_{24}) + x_{12}x_{34}(x_{23} - x_{14}) + x_{14}x_{34}(x_{23} + x_{24}) &= 0, \\
    x_{13}x_{23}(x_{14} + x_{24}) + x_{14}x_{24}(x_{13} + x_{23}) + x_{34}(x_{13}x_{23} - x_{14}x_{24}) &= 0, \\
    x_{12}(x_{13}x_{14} - x_{23}x_{24}) + x_{34}(x_{13}x_{23} - x_{14}x_{24}) &= 0.
\end{align*}
Lemma

The group $vP_3$ is 1-formal.

Proof: $vP_3 \cong N \ast \mathbb{Z}$, and $P_4 \cong N \times \mathbb{Z}$.

Lemma

The group $vP_4^+$ is not 1-formal.

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x_{12}(x_{13}x_{14} - x_{23}x_{24}) + x_{34}(x_{13}x_{23} - x_{14}x_{24}) &= 0.
\end{align*}

$\Rightarrow$ The group $vP_4^+$ is not 1-formal.
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Thank You!