1. Consider $\mathbb{R}^\infty$ equipped with the product topology (each factor $\mathbb{R}$ carries the Euclidean topology). Is $\mathbb{R}^\infty$ separable? Justify your answer.

2. A map $f : X \to Y$ is said to be a **local homeomorphism** if each point $x \in X$ has a neighborhood $U_x$ such that $f|_{U_x} : U_x \to f(U_x)$ is homeomorphism.

Let $X$ and $Y$ be path-connected compact Hausdorff spaces and let $f : X \to Y$ be a surjective local homeomorphism.

(a) Show that any point $y \in Y$ the set $f^{-1}(y)$ is finite.

(b) Show that for any two points $y_1, y_2 \in Y$ the sets $f^{-1}(y_1)$ and $f^{-1}(y_2)$ have the same cardinality.

3. Let $\mathcal{B}$ be the following collection of subsets of $\mathbb{R}$

$$
\mathcal{B} = \{ [a, b] \mid a, b \in \mathbb{R}, a < b \}
$$

(a) Show that $\mathcal{B}$ is a basis for a topology.

(b) With respect to the topology $\mathcal{T}_\mathcal{B}$ induced by $\mathcal{B}$, check whether or not $(\mathbb{R}, \mathcal{T}_\mathcal{B})$

(i) is connected,

(ii) is Hausdorff

(iii) is compact.

4. Show that a compact Hausdorff space is normal. This is an improvement of a theorem we discussed in class where we proved that a compact Hausdorff space is regular. Recall that a space $X$ is called normal if it is $T_2$ and $T_4$ (while it is regular if it is $T_1$ and $T_3$.)

5. Recall that we proved in class that if $A$ is a connected subspace of $X$ then $\overline{A}$ is also connected. Show that the analogous property fails to hold for path-connectedness. In other words, find an example of a path-connected space whose closure is not path-connected.

6. Consider the map $f : S^2 \to \mathbb{R}^4$ defined by

$$
f(x, y, z) = (x^2 - y^2, xy, xz, yz)
$$

Recall that $\mathbb{RP}^2$ can be defined as a quotient space of $S^2$ which identifies diametrically opposite points on $S^2$. Let $\pi : S^2 \to \mathbb{RP}^2$ be that quotient map.

(a) Show that there exists a map $\tilde{f} : \mathbb{RP}^2 \to \mathbb{R}^4$ such that $\tilde{f} \circ \pi = f$.

(b) Show that $\tilde{f} : \mathbb{RP}^2 \to \mathbb{R}^4$ is an embedding. Thus if we lived in 4-dimensional (Euclidean) space we could picture the real projective plane just fine.

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1Recall that a map $f : X \to Y$ is called an embedding if $f : X \to f(X)$ is a homeomorphism (where $f(X)$ is given the relative topology it inherits from $Y$).
7. Let $X$ be the union of the circles

$$X = \left\{ (x, y) \in \mathbb{R}^2 \mid \left( x - \frac{1}{n} \right)^2 + y^2 = \left( \frac{1}{n} \right)^2, \ n = 1, 2, 3, \ldots \right\}$$

topologized by the relative Euclidean topology (this space is referred to as the Hawaiian earrings). On the other hand, let $Y$ be the identification space of $\mathbb{R}$ (where $\mathbb{R}$ is given the Euclidean topology) gotten by collapsing all integers to a single point:

$$Y = \mathbb{R}/\mathcal{P}$$

with

$$\mathcal{P} = \{\ldots, -3, -2, -1, 0, 1, 2, 3\ldots\}, \{x\mid x \in \mathbb{R} - \mathbb{Z}\}$$

Show that $X$ and $Y$ are not homeomorphic (refer to figure 1 for a visual representation of $X$ and $Y$).

\[\text{Figure 1. A visual representation of } X \text{ and } Y. \text{ The upper figure only pictures the circles } (x - 1/n)^2 + y^2 = (1/n)^2 \text{ for } n = 1, 2, 3, 4, 5 \text{ (while } X \text{ is the union of infinitely many circles). The space } Y \text{ is gotten by collapsing all integers in } \mathbb{R} \text{ to a single point. The figure to the right shows 5 of infinitely many circles obtained by this construction.}\]