1. Let \( X \) be the set \( X = \{1, 2, 3\} \). List all possible topologies on \( X \). How many are there?

2. Let \( X \) be an infinite set and \( p \in X \) some arbitrary element in \( X \). Let \( T_{F,p} \) be the collection of subsets of \( X \) which either have finite complement or don’t contain the point \( p \). Show that \( T_{F,p} \) is a topology on \( X \) (called the Fort topology).

3. Let \( X \) be an infinite set and consider the collection \( \mathcal{T} \) of subsets of \( X \) defined as

\[
\mathcal{T} = \{ \emptyset, \text{All subsets } U \text{ of } X \text{ with infinite complement} \}
\]

Is \( \mathcal{T} \) a topology on \( X \)? Explain your answer.

4. Let \((X, \mathcal{T})\) be a topological space and let \( A \subseteq X \) be a subset of \( X \). We say that \( A \) is dense in \( X \) if for each non-empty \( U \in \mathcal{T} \) the intersection \( A \cap U \) is non-empty. For example the set \( A = X \) is always dense in \( X \). A topological space is said to be separable if it has a countable dense subset.

   (a) Show that a second countable\(^1\) topological space is automatically separable (the converse is not true in general).

   (b) Let \( T_\mathcal{E} \) be the Euclidean topology on \( \mathbb{R} \). Find a countable subset \( A \subset \mathbb{R} \) which is dense in \((\mathbb{R}, T_\mathcal{E})\) (you must show that your set \( A \) is in fact dense)\(^2\).

Solutions

1. There are 29 topologies on a set with 3 elements.

2. We need to check the 3 conditions from the definition of a topology.

   (a) \( \emptyset \in T_{F,p} \) since \( p \notin \emptyset \) and \( X \in T_{F,p} \) since the complement of \( X \) is finite.

   (b) Let \( U_i \in T_{F,p} \) for \( i \in \mathcal{I} \). We need to show that the union \( U = \bigcup_{i \in \mathcal{I}} U_i \) is also in \( T_{F,p} \). There are two cases to consider:

      (i) If \( p \notin U_i \) for all \( i \in \mathcal{I} \) then clearly \( p \notin U \) and so \( U \in T_{F,p} \).

\(^1\)Recall that a topological space \((X, \mathcal{T})\) is called second countable if it has a countable basis \( \mathcal{B} \).

\(^2\)Notice that the existence of a countable dense set \( A \) follows from part (a) of the exercise (remember we showed in class that \((\mathbb{R}^n, T_\mathcal{E})\) is second countable for all \( n \geq 1 \)). In fact, the solution to (a) along with having a concrete countable basis \( \mathcal{B} \) for \( T_\mathcal{E} \) provides a recipe for finding the set \( A \).
(ii) If \( p \in U_j \) for at least one index \( j \in I \) then \( X - U \) is finite. But \( X - U \subset X - U_j \) and a subset of a finite set is itself finite. Therefore \( U \in \mathcal{T}_{F,p} \).

(c) Given sets \( V_1, \ldots, V_n \in \mathcal{T}_{F,p} \), let \( V = \bigcap_{i=1}^n V_i \). We need to show that \( V \in \mathcal{T}_{F,p} \).

There are again two cases to distinguish:

(i) If \( p \notin V_j \) for some \( j \) in \( \{1, \ldots, n\} \) then clearly \( p \notin V \) and so \( V \in \mathcal{T}_{F,p} \).

(ii) If \( p \in V_i \) for all \( i = 1, \ldots, n \) then all the sets \( X - V_i \) are finite. But by DeMorgan’s laws we know that \( X - V = \bigcup_{i=1}^n (X - V_i) \). Since all the sets on the righthand side of this equality are finite, the lefthand side is also finite. Thus \( V \in \mathcal{T}_{F,p} \).

3. This is not a topology since \( X \) itself has finite complement and is not contained in \( \mathcal{T} \).

4. (a) Let \( \mathcal{B} = \{U_1, U_2, \ldots\} \) be a countable basis for \( X \). Pick an arbitrary point \( x_i \in U_i \) for all \( i = 1, 2, \ldots \). Define \( A \) to be

\[
A = \{x_1, x_2, \ldots\}
\]

Clearly \( A \) is countable since it has the same cardinality as \( \mathcal{B} \). To show that \( A \) is dense, pick any nonempty open set \( U \). Since \( \mathcal{B} \) is a basis, we can find a \( U_j \in \mathcal{B} \) such that \( U_j \subseteq U \). But then \( x_j \in A \cap U \) showing that \( A \cap U \neq \emptyset \).

(b) Recall that the following is a countable basis for \((\mathbb{R}, \mathcal{T}_E)\):

\[
\mathcal{B} = \{B_p(r) \mid p, r \in \mathbb{Q}, r > 0\}
\]

where as usual \( B_p(r) \) is the open ball centered at \( p \) and with radius \( r \). Using part (a) we can form \( A \) by picking a single element from each set in \( \mathcal{B} \). There are many choices here, all equally valid. A natural choice it to choose the point \( p \) itself from \( B_p(r) \). Since \( p \) is allowed to range through \( \mathbb{Q} \) we see that \( A = \mathbb{Q} \). This shows that the rational numbers are dense in \( \mathbb{R} \) (when we consider the Euclidean topology on \( \mathbb{R} \) but not necessarily with other topologies).