1. Let $X$ be the set $X = \{1, 2, 3\}$. List all possible topologies on $X$. How many are there?

2. Let $X$ be an infinite set and $p \in X$ some arbitrary element in $X$. Let $T_{F, p}$ be the collection of subsets of $X$ which either have finite complement or don’t contain the point $p$. Show that $T_{F, p}$ is a topology on $X$ (called the Fort topology).

3. Let $X$ be an infinite set and consider the collection $T$ of subsets of $X$ defined as

$$T = \{\emptyset, \text{All subsets } U \text{ of } X \text{ with infinite complement}\}$$

Is $T$ a topology on $X$? Explain your answer.

4. Let $(X, \mathcal{T})$ be a topological space and let $A \subseteq X$ be a subset of $X$. We say that $A$ is dense in $X$ if for each non-empty $U \in \mathcal{T}$ the intersection $A \cap U$ is non-empty. For example the set $A = X$ is always dense in $X$. A topological space is said to be separable if it has a countable dense subset.

   (a) Show that a second countable\footnote{Recall that a topological space $(X, \mathcal{T})$ is called second countable if it has a countable basis $\mathcal{B}$.} topological space is automatically separable (the converse is not true in general).

   (b) Let $\mathcal{T}_E$ be the Euclidean topology on $\mathbb{R}$. Find a countable subset $A \subset \mathbb{R}$ which is dense in $(\mathbb{R}, \mathcal{T}_E)$ (you must show that your set $A$ is in fact dense)\footnote{Notice that the existence of a countable dense set $A$ follows from part (a) of the exercise (remember we showed in class that $(\mathbb{R}^n, \mathcal{T}_E)$ is second countable for all $n \geq 1$). In fact, the solution to (a) along with having a concrete countable basis $\mathcal{B}$ for $\mathcal{T}_E$ provides a recipe for finding the set $A$.}.

5. Show that a function $f : \mathbb{R}^n \to \mathbb{R}$ is continuous if and only if for every closed set $A \subseteq \mathbb{R}$ the preimage $B = f^{-1}(A)$ is a closed subset of $\mathbb{R}^n$.

6. This exercise shows that a partial converse of problem 4 is true. Namely, let $(X, d)$ be a metric space and let $\mathcal{T}_d$ be the metric topology induced by $d$. Show that if $(X, \mathcal{T}_d)$ is separable then it is also second countable.

---

**Solutions**

1. There are 29 topologies on a set with 3 elements.
2. We need to check the 3 conditions from the definition of a topology.
(a) Let \( U \in T_{F,p} \) since \( p \notin \emptyset \) and \( X \in T_{F,p} \) since the complement of \( X \) is finite.
(b) Let \( U_i \in T_{F,p} \) for \( i \in I \). We need to show that the union \( U = \bigcup_{i \in I} U_i \) is also in \( T_{F,p} \). There are two cases to consider:
   (i) If \( p \notin U_i \) for all \( i \in I \) then clearly \( p \notin U \) and so \( U \in T_{F,p} \).
   (ii) If \( p \in U_j \) for at least one index \( j \in I \) then \( X - U_j \) is finite. But \( X - U \subset X - U_j \) and a subset of a finite set is itself finite. Therefore \( U \in T_{F,p} \).
(c) Given sets \( V_1, ..., V_n \in T_{F,p} \), let \( V = \bigcap_{i=1}^n V_i \). We need to show that \( V \in T_{F,p} \).
   There are again two cases to distinguish:
   (i) If \( p \notin V_j \) for some \( j \) in \( \{1, ..., n\} \) then clearly \( p \notin V \) and so \( V \in T_{F,p} \).
   (ii) If \( p \in V_i \) for all \( i = 1, ..., n \) then all the sets \( X - V_i \) are finite. But by DeMorgan’s laws we know that \( X - V = \bigcup_{i=1}^n (X - V_i) \). Since all the sets on the righthand side of this equality are finite, the lefthand side is also finite. Thus \( V \in T_{F,p} \).
3. This is not a topology since \( X \) itself has finite complement and is not contained in \( T \).
4. (a) Let \( \mathcal{B} = \{U_1, U_2, ...\} \) be a countable basis for \( X \). Pick an arbitrary point \( x_i \in U_i \) for all \( i = 1, 2, .... \). Define \( A \) to be
   \[
   A = \{x_1, x_2, ...
   \]
   Clearly \( A \) is countable since it has the same cardinality as \( \mathcal{B} \). To show that \( A \) is dense, pick any nonempty open set \( U \). Since \( \mathcal{B} \) is a basis, we can find a \( U_j \in \mathcal{B} \) such that \( U_j \subseteq U \). But then \( x_j \in A \cap U \) showing that \( A \cap U \neq \emptyset \).
   (b) Recall that the following is a countable basis for \( (\mathbb{R}, \mathcal{T}_E) \):
   \[
   \mathcal{B} = \{B_p(r) \mid p, r \in \mathbb{Q}, r > 0\}
   \]
   where as usual \( B_p(r) \) is the open ball centered at \( p \) and with radius \( r \). Using part (a) we can form \( A \) by picking a single element from each set in \( \mathcal{B} \). There are many choices here, all equally valid. A natural choice it to choose the point \( p \) itself from \( B_p(r) \). Since \( p \) is allowed to range through \( \mathbb{Q} \) we see that \( A = \mathbb{Q} \). This shows that the rational numbers are dense in \( \mathbb{R} \) (when we consider the Euclidean topology on \( \mathbb{R} \) but not necessarily with other topologies).
5. Assume that \( f: \mathbb{R}^n \to \mathbb{R} \) is continuous. Let \( B \subseteq \mathbb{R} \) be closed set and define \( A \) to be \( f^{-1}(B) \). An easy check reveals that
   \[
   A = \mathbb{R}^n - f^{-1}(\mathbb{R} - B)
   \]
   Since \( f \) is continuous, the set \( f^{-1}(\mathbb{R} - B) \) is an open set (since \( \mathbb{R} - B \) is an open set) and so \( A \) is closed.
6. Let \( A \) be a dense countable set in \( X \) and define \( \mathcal{B} \) as
   \[
   \mathcal{B} = \{B_x(r) \mid x \in A, r \in \mathbb{Q}, r > 0\}
   \]
where $B_{x}(r) = \{ y \in X | d(x, y) < r \}$. Since $A$ and $\mathbb{Q}$ are countable then so is $\mathcal{B}$. We claim that $\mathcal{B}$ is a basis for the metric topology $\mathcal{T}_{d}$. We start proving this by first showing that $\mathcal{B}$ is the basis of some topology.

**Claim 1:** $\mathcal{B}$ is the basis of some topology.

*Proof.* (Of claim 1) We check that the two conditions of being a basis are met.

(a) Given any point $p \in X$ we need to show that there is an element $U \in \mathcal{B}$ such that $p \in U$. Consider the open set $B_{p}(1)$. This is an open set and so the intersection $A \cap B_{x}(1) \neq \emptyset$. Pick an element $a \in A \cap B_{x}(1)$. Then $d(a, p) < 1$ and so $p \in B_{a}(1) \in \mathcal{B}$.

(b) Given two sets $U_{1}, U_{2} \in \mathcal{B}$ and a point $p \in U_{1} \cap U_{2}$ we need to show that there is a third set $U_{3} \in \mathcal{B}$ such that $p \in U_{3} \subseteq U_{1} \cap U_{2}$. Since $U_{1} \cap U_{2}$ is an open set, we can find an $\varepsilon > 0$ such that $B_{p}(2\varepsilon) \subseteq U_{1} \cap U_{2}$. But clearly $B_{p}(\varepsilon) \subseteq B_{p}(2\varepsilon) \subseteq U_{1} \cap U_{2}$. Since $B_{p}(\varepsilon)$ is a nonempty open set, we can find an element $a \in A \cap B_{p}(\varepsilon)$. Let $\rho$ be any rational number with $d(p, a) < \rho < \varepsilon$ and let $U_{3}$ be the set $B_{a}(\rho) \in \mathcal{B}$. Since $d(p, a) < \rho$ we see that $p \in U_{3}$. But on the other hand, given any point $y \in U_{3}$, by the triangle inequality for $d$ we get

$$d(y, x) \leq d(y, a) + d(a, x) < \rho + \varepsilon < 2\varepsilon$$

Therefore $y \in B_{x}(2\varepsilon) \subseteq U_{1} \cap U_{2}$. Since this is true for all $y \in U_{3}$ we see that indeed $U_{3} \subseteq U_{1} \cap U_{2}$. This completes the proof of claim 1.

□

**Claim 2:** The topology generated by $\mathcal{B}$ is the metric topology $\mathcal{T}_{d}$.

*Proof.* (Of claim 2) Let $\mathcal{T}_{B}$ be the topology generated by $\mathcal{B}$. Since $\mathcal{B} \subseteq \mathcal{T}_{d}$ it follows that $\mathcal{T}_{B} \subseteq \mathcal{T}_{d}$. It remains to see the converse. For that purpose, pick any open set $U \in \mathcal{T}_{d}$. We need to show that $U$ also belongs to $\mathcal{T}_{B}$. In other words, we need to show that $U$ is a union of sets belonging to $\mathcal{B}$.

Let $p \in U$ be any point. Since $U$ is open we can find an $\varepsilon > 0$ so that $B_{p}(2\varepsilon) \subseteq U$. Pick a point $a \in A \cap B_{p}(\varepsilon)$ and let $U_{p} = B_{a}(\varepsilon)$. Since $d(a, p) < \varepsilon$ we see that $p \in U_{p}$. Given any other element $y \in U_{p}$, using the triangle inequality again we find that

$$d(y, p) \leq d(y, a) + d(a, p) < \varepsilon + \varepsilon = 2\varepsilon$$

showing that $y \in B_{x}(2\varepsilon) \subseteq U$. Thus $U_{p} \subseteq U$. Doing this procedure for each $p \in U$ we get

$$U = \bigcup_{p \in U} U_{p} \quad U_{p} \in \mathcal{B}$$

showing that $U \in \mathcal{T}_{B}$. This completes the proof of claim 2. □

In conclusion, the claims 1 and 2 show that $\mathcal{B}$ is a countable basis for $\mathcal{T}_{d}$ and so $(X, \mathcal{T}_{d})$ is a second countable topological space.