1. Let $X$ be a topological space and $A$ a subset of $X$. Show that

$$\overline{X - A} = X - \text{Int}(A) \quad \text{and} \quad \text{Int}(X - A) = X - \overline{A}$$

Use these equalities to show that $\partial A = \partial (X - A)$.

2. Let $(X, <)$ be a simply ordered set and let $\mathcal{T}_\sigma$ be the order topology on $X$. Show that with respect to the order topology the inclusion $\langle a, b \rangle \subseteq [a, b]$ holds for any two elements $a, b \in X$ with $a < b$. Under what conditions is the equality $\langle a, b \rangle = [a, b]$ true?

3. For subsets $A \subseteq X$ and $B \subseteq Y$ of the topological spaces $X$ and $Y$, show that $A \times B = \overline{A} \times \overline{B}$ in the product topology on $X \times Y$.

4. Show that $X$ is a Hausdorff space if and only if the diagonal $\Delta = \{(x, x) \mid x \in X\}$ is a closed set in $X \times X$ (equipped with the product topology).

5. Let $X$ be a topological space and let $A_i, i \in I$ be a family of subsets of $X$ with $I$ some (possibly infinite) indexing set. Prove or disprove the equality

$$\bigcup_{i \in I} A_i = \overline{\bigcup_{i \in I} A_i}$$

6. Which of the separation axioms $T_0 - T_4$ are satisfied by the Fort topology $\mathcal{T}_{F,p}$ on $X = \mathbb{R}$? See the Notes 2 online or Homework 1 for a definition of the Fort topology. The particular choice of the point $p$ is immaterial for this problem, choose it as you please.

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**Solutions**

1. (a) $\overline{X - A} = X - \text{Int}(A)$

   This follows from the definitions of the closure and interior of the set $A$. Let $\mathcal{B}$ and $\mathcal{C}$ be the collections of subsets of $X$ defined by

   $\mathcal{B} = \{ B \subseteq X \mid X - A \subseteq B, B \text{ is a closed set} \}$

   $\mathcal{C} = \{ C \subseteq X \mid C \subseteq A, C \text{ is an open set} \}$
Observe that there is a bijective correspondence between these two collection, the one which associates to a set $C \in \mathcal{C}$ to the set $B = X - C \in \mathcal{B}$. On the other hand

$$\overline{X - A} = \bigcap_{B \in \mathcal{B}} B$$

and

$$\text{Int}(A) = \bigcup_{C \in \mathcal{C}} C$$

Therefore, using DeMorgan’s laws, we have

$$X - \text{Int}(A) = X - \bigcup_{C \in \mathcal{C}} C = \bigcap_{C \in \mathcal{C}} (X - C) = \bigcap_{B \in \mathcal{B}} B = \overline{X - A}$$

(b) $\text{Int}(X - A) = X - \overline{A}$

Follows by the same principle as part (a). Let $\mathcal{D}$ and $\mathcal{E}$ be the collections

$$\mathcal{D} = \{ D \subseteq X \mid D \subseteq X - A, D \text{ is an open set} \}$$

$$\mathcal{E} = \{ E \subseteq X \mid A \subseteq E, E \text{ is a closed set} \}$$

These are again bijective sets under the function $D \mapsto X - E$. As in part (a) a simple use of DeMorgan’s laws completes the job:

$$X - \overline{A} = X - \bigcap_{E \in \mathcal{E}} E = \bigcup_{E \in \mathcal{E}} (X - E) = \bigcup_{D \in \mathcal{D}} D = \text{Int}(X - A)$$

(c) This part is an exercise in set theory:

$$\partial(X - A) = \overline{X - A} - \text{Int}(X - A)$$

$$= (X - \text{Int}(A)) - (X - \overline{A})$$

$$= \overline{A} - \text{Int}(A)$$

The fact that $(X - \text{Int}(A)) - (X - \overline{A}) = \overline{A} - \text{Int}(A)$ can easily be checked directly but becomes obvious from the picture below.

![Diagram](image)

**Figure 1.** The big oval in each of the 3 pictures is the set $X$. The dotted line indicates the set $\text{Int}(A)$ and the polygon encloses $\overline{A}$. The set $A$ itself is not visible. The shaded region in each picture represents the set listed under the diagram. The difference of the first two sets clearly equals the last set.

2. Since $[a, b] = (-\infty, a) \cup (b, \infty)$, the latter of which is an open set in the order topology, it follows that $[a, b]$ is a closed set. On the other hand $\langle a, b \rangle \subseteq [a, b]$ and therefore $\overline{\langle a, b \rangle} \subseteq [a, b]$ (because the closure is the smallest closed set containing $\langle a, b \rangle$).
For the other part of the question, notice that \([a, b] = \langle a, b \rangle \cup \{a, b\}\). Thus the only possibilities for \(\langle a, b \rangle\) are

\[
\langle a, b \rangle = \begin{cases} 
[a, b] = \langle a, b \rangle \cup \{a, b\} \\
[a, b] = \langle a, b \rangle \cup \{a\} \\
[a, b] = \langle a, b \rangle \cup \{b\} \\
[a, b]
\end{cases}
\]

Each of the 4 cases above can occur. If \(\langle a, b \rangle = [a, b]\) then \([a, b]\) is a closed set and so \(X - [a, b] = (-\infty, a) \cup [b, \infty)\) is an open set. This for example happens if there is an element \(c \in X\), \(c < b\) such that there is no element \(x \in X\) with \(c < x < b\). In that case \([b, \infty) = \langle c, \infty \rangle\).

Similarly if there is an element \(d \in X\), \(a < d\) such that there is no \(x \in X\) with \(a < x < d\) then \(\langle a, b \rangle\) is a closed set. If both \(c\) and \(d\) exist then \(\langle a, b \rangle\) is itself a closed set.

3. This was shown in class.

4. Suppose \(X\) is Hausdorff. We will show that \(\Delta\) is closed in \(X \times X\) by showing that \(X \times X - \Delta\) is an open set. Pick an arbitrary point \((a, b) \in X \times X - \Delta\). Then \(a \neq b\) (since otherwise \((a, b) \in \Delta\)) and so the Hausdorff property guarantees the existence of open sets \(U_a, V_b\) with \(a \in U_a\), \(b \in V_b\) and \(U_a \cap V_b = \emptyset\). These properties in turn imply that \((a, b) \in U_a \times V_b\) and \(U_a \times V_b \subset X \times X - \Delta\). Since \((a, b)\) was arbitrary we can write

\[
X \times X - \Delta = \bigcup_{(a, b) \in X \times X - \Delta} U_a \times V_b
\]

The sets on the right-hand side are open sets in the product topology.

Assume that \(\Delta\) is closed in \(X \times X\). To show that \(X\) is Hausdorff, pick two arbitrary points \(a, b \in X\) \((a \neq b)\). We need to find to open, disjoint sets \(U, V\) with \(a \in U\), \(b \in V\). Since \(\Delta\) is closed, \(X \times X - \Delta\) is open and is therefore a union of open sets:

\[
X \times X - \Delta = \bigcup_i U_i \times V_i
\]

where \(U_i\), \(V_i\) is some family of open subsets of \(X\). Find the index \(j\) (there may be many such indices but there is for sure at least one!) for which \((a, b) \in U_j \times V_j\). Then \(a \in U_j\) and \(b \in V_j\). If \(c \in U_j \cap V_j\) then \((c, c) \in (U_j \times V_j) \cap \Delta\) which is impossible since \(U_j \times V_j \subset X \times X - \Delta\). Thus \(U_j \cap V_j = \emptyset\) as needed.

5. Since for each \(j \in \mathcal{I}\) we have the inclusion \(A_j \subseteq \bigcup_i A_i\) and the latter is a closed set, it must be that the inclusion \(\overline{A_j} \subseteq \bigcup_i A_i\) also holds. Taking the union over \(j\) gives \(\overline{\bigcup_i A_i} \subseteq \bigcup_i \overline{A_i}\).

On the other hand, the inclusion \(\bigcup_i A_i \subseteq \bigcup_i \overline{A_i}\) may fail to be true. For example, consider the Euclidean topology on \(\mathbb{R}\) and let \(A_i\) be the sets

\[
A_i = \left[ 0, 2 - \frac{1}{i} \right]
\]

These are all closed sets and so \(\overline{A_i} = A_i\). The union of these sets is \(\bigcup_i A_i = \bigcup_i A_i = [0, 2]\) and so

\[
\overline{\bigcup_i A_i} = [0, 2] \not\subset [0, 2) = \bigcup_i \overline{A_i}
\]
6. • The Fort topology is Hausdorff: Given two points $a, b \in \mathbb{R}$ we need to find two disjoint open sets $U, V$ with $a \in U$ and $b \in V$. There are two cases to consider
(a) If $a = p$ let $V = \{b\}$ and $U = \mathbb{R} - \{b\}$. If $b = p$ let $U = \{a\}$ and $V = \mathbb{R} - \{a\}$.
(b) If both $a \neq p$ and $b \neq p$ then let $U = \{a\}$ and $V = \{b\}$.

• The Fort topology is $T_4$: Given two disjoint closed subsets $A, B$ of $X$, we need to find disjoint open subsets $U, V$ of $X$ with $A \subseteq U$ and $B \subseteq V$. Notice that closed sets come in two flavours – either they contain $p$ or they are finite sets. According to which flavor $A$ and $B$ are, there are again two cases.
(a) If $p \in A$ then $B$ must be finite and $p \notin B$. Take $U = \mathbb{R} - B$ and $V = B$. The case $p \in B$ is similar.
(b) If $p \notin A$ and $p \notin B$ take $U = A$ and $V = B$.

Since $T_4$ implies $T_3$ and $T_2$ implies $T_1$ and $T_0$ it follows that the Fort topology on $\mathbb{R}$ is $T_0 - T_4$. 