1. Let $\pi : X \to Y$ be a quotient map. Show that if $Y$ is connected and for each $y \in Y$ the subspace $\pi^{-1}(\{y\})$ is connected then $X$ is connected as well.

2. Let $X$ be a topological space with the property that each $x \in X$ has a neighborhood homeomorphic to $\mathbb{R}^n$ with the Euclidean topology. Show that if $X$ is connected then it is also path-connected.

3. Let $X = \mathbb{R}$ be given the Fort topology $T_{F,p}$ with $p = 0$. Verify whether $X$ is
   (a) connected,
   (b) path-connected,
   (c) compact.

4. Let $f : X \to Y$ be a closed map and assume that for each $y \in Y$ the preimage $f^{-1}(\{y\})$ is a compact subspace of $X$. Show that then for every compact subset $K \subseteq Y$ the preimage $f^{-1}(K)$ is also a compact subspace of $X$.

\textbf{Solutions}

1. Suppose to the contrary that $X$ is not connected and let $X = A \cup B$ be a separation of $X$. Let $A' = \pi(A)$ and $B' = \pi(B)$. We will show that this is a separation of $Y$ – a contradiction showing that $X$ must be connected.

   It is clear that $A', B' \neq \emptyset$ since $A, B \neq \emptyset$ and $Y = A' \cup B'$ since $\pi$ is surjective. Furthermore, $A' \cap B' = \emptyset$ since for any $y \in Y$ we must have $\pi^{-1}(y) \subseteq A$ or $\pi^{-1}(y) \subseteq B$ by connectedness of $\pi^{-1}(y)$. It remains to show that $A'$ and $B'$ are open. Towards this goal consider $\pi^{-1}(A')$. Clearly $A \subseteq \pi^{-1}(A')$ and to see equality, observe that if $x \in B \cap \pi^{-1}(A')$ then $\pi(x) \in B' \cap A' = \emptyset$. Thus $A = \pi^{-1}(A')$ and since $A$ is open and $\pi$ is a quotient map then $A'$ is also open. Similarly one sees that $B'$ is also open.

2. Pick an arbitrary $x_0 \in X$ and let $U_0 = \{x \in X | x \text{ is connected to } x_0 \text{ by a path } \}$. Then $U$ is nonempty (since $x_0 \in U_0$) and open since given an $x \in U_0$ the neighborhood $V_x$ of $x$ homeomorphic to $\mathbb{R}^n$ is contained in $U_0$: This is so since given any point $y \in V_x$ there is a path $\alpha : [0,1] \to X$ connecting $y$ to $x$. Since $x$ is in $U_0$ there must also be
a path $\beta: [0, 1] \to X$ connecting $x$ to $x_0$. But then
\[ \gamma: [0, 1] \to X : t \mapsto \begin{cases} \alpha(2t) & ; t \in [0, \frac{1}{2}] \\ \beta(2t - 1) & ; t \in [\frac{1}{2}, 1] \end{cases} \]
is a path connecting $y$ to $x_0$ and therefore $y \in U_0$.

Thus $U_0$ is open since $U_0 = \bigcup_{x \in U_0} V_x$. On the other hand, $U_0$ is also closed since its complement is open: To see that, suppose that $x \in X - U_0$ and pick $y \in V_x$. Since $y$ can be connected to $x$ by a path, if there were a path from $y$ to $x_0$ then we could also connect $x$ to $x_0$ which is not possible for $x \notin U_0$. Thus $V_x \subseteq X - U_0$ for all $x \in X - U_0$. But then again $X - U_0 = \bigcup_{x \in X - U_0} V_x$ which is open.

Thus $X = U_0 \cup (X - U_0)$ is a union of two open and disjoint sets. Since $X$ is connected, one of these sets must be empty. Since $U \neq \emptyset$ we infer that $X - U_0$ is empty and so $X = U_0$. Since $U_0$ is path connected so is $X$.

3. (a) $X$ is not connected since $A = \mathbb{R} - \{1\}$ and $B = \{1\}$ are a separation of $X$.
   (b) $X$ cannot be path connected since it is not connected.
   (c) $X$ is compact: Given any open cover $\mathcal{C} = \{C_i \mid i \in \mathcal{I}\}$ there must be an index $i_0 \in \mathcal{I}$ such that $0 \in C_{i_0}$. But then $\mathbb{R} - C_{i_0}$ is finite and hence covered by finitely many sets $\{C_{i_1}, ..., C_{i_k}\}$. Then $\mathcal{C}' = \{C_{i_0}, C_{i_1}, ..., C_{i_k}\}$ is a finite subcover of $\mathcal{C}$.

4. Pick a compact set $K \subseteq Y$ and let $\mathcal{C} = \{C_i \mid i \in \mathcal{I}\}$ be an open cover of $f^{-1}(K)$. For each $y \in K$, the set $f^{-1}(y)$ is compact and can therefore be covered by a finite number of elements from the cover $\mathcal{C}$. Let $U_y$ be the union of this finite collection of open sets. If we can show that $f^{-1}(K)$ can be covered with finitely many $U_y$’s then $f^{-1}(K)$ is compact.

We know that $U_y$ is an open set and $f^{-1}(y) \subseteq U_y$. Let $V_y$ be the set $Y - f(X - U_y)$. Since $U_y$ is open, $X - U_y$ is closed and since $f$ is a closed map the set $f(X - U_y)$ is closed. But then its complement (which is equal to $V_y$) must be open. Furthermore, $y \in V_y$ since otherwise we would have $y \in f(X - U_y)$. This would imply that $y = f(x)$ for some $x \in X - U_y$. But then $x \notin U_y$ and $x \in f^{-1}(y)$ which is a contradiction. Thus $V_y$ is a neighborhood of $y$ for each $y \in K$.

Let $\mathcal{D} = \{V_y \mid y \in K\}$, this is an open cover of $K$ which by compactness of $K$ has a finite subcover $\mathcal{D}' = \{V_{y_1}, ..., V_{y_n}\}$. Now since $K \subseteq \bigcup_{i=1}^n V_{y_i}$ we get $f^{-1}(K) \subseteq \bigcup_{i=1}^n f^{-1}(V_{y_i}) \subseteq \bigcup_{i=1}^n U_{y_i}$ and each $U_{y_i}$ is covered by finitely many elements from $\mathcal{C}$. 