1. Let $X$ be a second countable topological space and let $C$ be any open cover of $X$. Show that there is a countable cover $D = \{D_1, D_2, D_3, \ldots\}$ with the property that for each $D_i \in D$ there is some $C \in C$ with $D_i \subseteq C$ (this was a crucial step in proving the alternative characterizations of compactness on metric spaces we did in class).

2. Let $X$ be a compact space and let $A_i \subseteq X$ be a nested sequence of nonempty closed subsets of $X$, i.e.

$$A_1 \supseteq A_2 \supseteq A_3 \supseteq A_4 \supseteq \ldots$$

Show that the intersection $\cap_{i=1}^{\infty} A_i$ cannot be the empty set.

3. Show that if $(X, d)$ is a compact metric space then it has finite diameter. Furthermore, show that there exist points $x_0, y_0 \in X$ such that $\text{diam}(X) = d(x_0, y_0)$. Recall that the diameter of $X$ is defined as

$$\text{diam}(X) = \sup\{d(x, y) \mid x, y \in X\}$$

4. Show that $[0, 1]^\infty$ in the box topology is not a compact metric space. While we have learned in class that $([0, 1]^\infty, T_{\text{box}})$ is not a metric space, we haven’t proved that so you shouldn’t use this fact in your proof.

Solutions

1. Let $B = \{B_1, B_2, \ldots\}$ be a countable basis for the topology on $X$. Given an open cover $\mathcal{C} = \{C_j \mid j \in J\}$ of $X$ define $\mathcal{D}$ to be the collection of open sets

$$\mathcal{D} = \{B_k \in B \mid \text{there exists some } C_j \in \mathcal{C} \text{ with } B_k \subseteq C_j\}$$

Since $\mathcal{D}$ is a subset of $\mathcal{B}$ which is a countable set, $\mathcal{D}$ is also countable. Every element of $\mathcal{D}$ is open and contained in some element of $\mathcal{C}$. To complete the problem we need to see that $\mathcal{D}$ is a cover of $X$. Given any point $x \in X$, find a set $C_j \in \mathcal{C}$ with $x \in C_j$. As $C_j$ is an open set, we can write it as a union of basis elements. But then there must be a basis element which contains $x$ (and is itself contained in $C_j$) showing that every $x \in X$ is contained in some $B_k$ from $\mathcal{D}$. Thus $\mathcal{D}$ is the desired cover.

2. Consider the open sets $U_i = X - A_i$. If $\cap_{i=1}^{\infty} A_i = \emptyset$ then

$$X = X - \emptyset = X - \cap_{i=1}^{\infty} A_i = \cup_{i=1}^{\infty} (X - A_i) = \cup_{i=1}^{\infty} U_i$$
which shows that \( \mathcal{U} = \{ U_1, U_2, \ldots \} \) is an open cover of \( X \). Since \( X \) is compact there must be a finite subcover \( \mathcal{U}' = \{ U_{i_1}, \ldots, U_{i_m} \} \) of \( \mathcal{U} \). Without loss of generality we can assume that \( i_j < i_k \) for \( j < k \). Then we have

\[
X = \bigcup_{j=1}^m U_{i_j} = \bigcup_{j=1}^m (X - A_{i_j}) = X - \bigcap_{j=1}^m A_{i_j} = X - A_{i_m}
\]

which implies that \( A_{i_m} \) is the empty set, contrary to assumption.

3. Give \( X \times X \) the product topology and consider the function \( d : X \times X \to \mathbb{R} \) (where carries the Euclidean metric). We will first show that \( d \) is continuous by showing it is continuous at each point \((x_0, y_0) \in X \times Y\). Pick a neighborhood \( V = \langle d(x_0, y_0) - \varepsilon, d(x_0, y_0) + \varepsilon \rangle \) of \( d(x_0, y_0) \) (where \( \varepsilon > 0 \) is arbitrary). We need to find an open set \( U \subseteq X \times Y \) so that \( d(U) \subseteq V \). The open set \( U = B_{x_0}(\varepsilon/2) \times B_{y_0}(\varepsilon/2) \) has this property for if \( x \in B_{x_0}(\varepsilon/2) \) and \( y \in B_{y_0}(\varepsilon/2) \) then

\[
d(x, y) \leq d(x, x_0) + d(x_0, y_0) + d(y_0, y) < d(x_0, y_0) + \varepsilon \implies d(x, y) - d(x_0, y_0) < \varepsilon
\]

which when combined gives \( |d(x, y) - d(x_0, y_0)| < \varepsilon \) showing that \( d(x, y) \in V \) for each \((x, y) \in U\).

Since \( X \) is compact then so is \( X \times X \) and since \( d \) is continuous it attains a maximum on \( X \times X \), say at \( (x_{\max}, y_{\max}) \in X \times X \). Thus

\[
\text{diam}X = d(x_{\max}, y_{\max}) < \infty
\]
as desired.

4. Consider the sequence \( x_n = (0, \ldots, 0, 1, 0, 0, \ldots) \) where 1 is placed into the \( n \)-th slot of the sequence. Clearly \( x_n \in [0, 1]^\infty \). If the latter were a compact metric space then there would have to be a convergent subsequence of \( \{x_n\} \). Let \( U_n \) be the open sets defined as

\[
U_n = \langle -\frac{1}{2}, \frac{1}{2} \rangle \times \ldots \times \langle -\frac{1}{2}, \frac{1}{2} \rangle \times \langle \frac{1}{2}, \frac{3}{2} \rangle \times \langle \frac{1}{2}, \frac{1}{2} \rangle \times \ldots
\]

where \( \langle \frac{1}{2}, \frac{3}{2} \rangle \) is the \( n \)-th factor. Then \( x_n \in U_n \) and \( x_m \notin U_n \) for all \( m \neq n \). Thus \( \{x_n\} \) cannot have a convergent subsequence.