1. The One Point Compactification

**Definition 1.1.** A compactification of a topological space \( X \) is a compact topological space \( Y \) containing \( X \) as a subspace.

Given any non-compact space \( X \), compactifications always exist. This section explores the smallest possible compactification obtained by adding a single point to \( X \) and extending the topology in a suitable way. The thus obtained compactification of \( X \) is called the **one-point compactification** of \( X \). Here are the details:

Given any non-compact space \( X \), define \( Y = X \cup \{p\} \) where \( p \) is some abstract point (and \( p \not\in X \)). Let \( T_X \) be the existent topology on \( X \). We define a new topology \( T_Y \) on \( Y \) as follows: A subset \( U \) of \( Y \) is open if either

1. \( p \not\in U \) and \( U \in T_X \) or
2. \( p \in U \) and \( X - U \) is a compact closed subset of \( X \).

**Lemma 1.2.** The collection \( T_Y \) of subsets of \( Y \) is a topology.

**Proof.** The proof comes down to checking the 3 properties of being a topology.

1. Clearly \( \emptyset \in T_Y \) since \( \emptyset \in T_X \subset T_Y \). On the other hand \( Y \in T_Y \) since \( Y = X \cup \{p\} \) and \( X - Y = \emptyset \) which is closed and compact.
2. Let \( U_i \in T_Y \) for \( i \in I \) and let \( U = \bigcup_{i \in I} U_i \). We need to show that \( U \) lies in \( T_Y \).
   - If \( p \not\in U \) then all the sets \( U_i \) are in \( T_X \) and we’re done. If \( p \in U \) then we can decompose \( I \) as \( I = I_0 \sqcup I_1 \) so that
     - (a) If \( i \in I_0 \) then \( p \not\in U_i \).
     - (b) If \( i \in I_1 \) then \( p \in U_i \). In this case let’s write \( U_i = A_i \cup \{p\} \) with \( A_i \subset X \) such that \( X - A_i \) is compact and closed.
   
   But then \( U = A \cup \{p\} \) with
   
   \[ A = (\bigcup_{i \in I_0} U_i) \cup (\bigcup_{i \in I_1} A_i) \]

   From DeMorgan’s law we then find
   
   \[ X - A = X - \left[ (\bigcup_{i \in I_0} U_i) \cup (\bigcup_{i \in I_1} A_i) \right] = (\bigcap_{i \in I_0} (X - U_i)) \cap (\bigcap_{i \in I_1} (X - A_i)) \]

   which is a closed subset of \( X - A \) which in turn is compact. Thus \( X - A \) is itself compact and so \( U \) is in \( T_Y \) by definition.
3. Pick \( V_1, \ldots, V_n \in T_Y \) and set \( V = \bigcap_{i=1}^{n} V_i \). If \( p \in V \) then each \( V_i \) has the form \( V_i = A_i \cup \{p\} \) with \( X - A_i \) being compact and closed. Therefore \( V = A \cup \{p\} \) with \( A = \bigcap_{i=1}^{n} A_i \). To see that \( V \) is open we need to check that \( X - A \) is closed and compact:

   \[ X - A = X - \bigcap_{i=1}^{n} A_i = \bigcup_{i=1}^{n} (X - A_i) \]

   The latter expression is a finite union of closed compact sets and is therefore itself closed and compact.

   If \( p \not\in V \) then at least one \( V_i \) contains \( p \). Up to re-ordering the summands \( V_i \) we can then write

   \[ V_i \in T_X \quad \text{and} \quad V_j = A_j \cup \{p\} \]
for $i = 1, \ldots, m$ and $j = m + 1, \ldots, n$. Here $X - A_j$ is compact and closed, in particular $A_j$ is open in $X$. But then
\[ V = \bigcap_{i=1}^n V_i = (\bigcap_{i=1}^m V_i) \cap (\bigcap_{j=m+1}^n A_j) \]
which is a finite intersection of open sets in $X$ and therefore an open set in $X$.

\[ \square \]

**Lemma 1.3.** With the notation as above, $X$ is a dense subspace of $Y$.

*Proof.* The lemma contains two claims – that $X$ is dense in $Y$ and that the subspace topology on $X$ induced by $T_Y$ coincides with $T_X$. The latter is a mere observation: Writing $T_Y|_X$ for the induced subspace topology on $X$ we find that
\[ T_Y|_X = \{ U \cap X | U \in T_Y \} \]
\[ = \{ U \cap X | U \in T_X \text{ or } U = A \cup \{p\} \text{ with } X - A \text{ closed and compact} \} \]
\[ = \{ U \cap X | U \in T_X \} \cup \{ U \cap X | U = A \cup \{p\} \text{ with } X - A \text{ closed and compact} \} \]
\[ = T_X \cup \{ A | X - A \text{ is compact and } A \text{ is open} \} \]
\[ = T_X \]
Thus is remains to see that $X$ is dense in $Y$. This is equivalent to showing that $X = Y$. If the latter were not true then we would be forced to conclude that $X = Y$. But then $Y - X = \{p\}$ would have to be an open set. Since it has the form $\emptyset \cup \{p\}$ we would need $X - \emptyset = X$ to be compact and closed. But our assumption was that $X$ is not compact. Thus we must have $X = Y$. \[ \square \]

**Lemma 1.4.** Continuing with the notation as above, $Y$ is a compact space.

*Proof.* Let $\mathcal{F}$ be an open cover of $Y$. There must be some set $U_0 \in \mathcal{F}$ which contains $p$. But then $X - U_0$ is a compact set covered by $\mathcal{F}$ and so there is a finite number of elements $U_1, \ldots, U_n \in \mathcal{F}$ whose union contains $X - U_0$. Then $\mathcal{F}' = \{ U_0, U_1, \ldots, U_n \}$ is a finite subcover of $\mathcal{F}$ showing that $Y$ is compact. \[ \square \]

We summarize the last 3 lemmas in the following theorem.

**Theorem 1.5 (One point compactification).** Let $X$ be a non-compact space and let $Y = X \cup \{p\}$ for some abstract point $p \notin X$. Define the topology $T_Y$ on $Y$ as
\[ T_Y = T_X \cup \{ A \cup \{p\} | X - A \text{ is closed and compact } \} \]
Then $Y$ is compact and $X$ is a dense subspace of $Y$. The space $Y$ is called the one-point compactification of $X$\(^{1}\).

2. **Exercises**

(1) Show that the one-point compactification of $\mathbb{R}^n$ (with the Euclidean topology) is homeomorphic to the $n$-sphere
\[ S^n = \{ x \in \mathbb{R}^{n+1} | |x| = 1 \} \]
equipped with the relative topology.

\(^1\)If $X$ is itself compact then the theorem is still true except that $X$ is no longer dense in $Y$.\]
(2) Define a “two-point compactification” of a non-compact space $X$. Show that the two-point compactification of $S^1 \times [0,1]$ is again homeomorphic to $S^2$. 