Quiz 3

1. (10 points) For each \( n \in \mathbb{N} \) let \( x_n = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} \). Show that \( x_n \) is not a Cauchy sequence.

Solution: Let \( m = n + k \) and consider \( |x_m - x_n| \):

\[
|x_m - x_n| = \left| \left( \frac{1}{1} + \frac{1}{2} + \ldots + \frac{1}{n+k} \right) - \left( \frac{1}{1} + \frac{1}{2} + \ldots + \frac{1}{n} \right) \right| \\
= \left| \frac{1}{n+1} + \frac{1}{n+2} + \ldots + \frac{1}{n+k} \right| \\
> \frac{k}{n+k}
\]

With \( n \) fixed and with \( k \to \infty \) the expression \( k/(n+k) \) converges to 1. If the sequence \( x_n \) were Cauchy then for \( \varepsilon = 1/2 \) there would have to be some \( n_0 \in \mathbb{N} \) such that \( n \geq n_0 \) and \( k \geq 0 \) would imply that \( |x_{n+k} - x_n| < 1/2 \). This however contradicts the fact that for \( n \) fixed the limit \( \lim_{k \to \infty} |x_{n+k} - x_n| = 1 \). Therefore \( x_n \) cannot be Cauchy.

An alternative solutions is to observe that \( x_n \) is not convergent (since it is unbounded as shown in class) but Cauchy sequences in \( \mathbb{R} \) are convergent.

2. (10 points) Using the definition of a limit at infinity, show that \( \lim_{n \to \infty} (6n^2 + n) = \infty \).

Solution: By definition of the limit at infinity, we need to show that for any \( M \in \mathbb{R} \) there is some \( n_0 \in \mathbb{N} \) such that \( n \geq n_0 \) implies \( 6n^2 + n \geq M \). Since \( 6n^2 + n > n \), any \( n_0 \geq M \) will do.