CHAPTER 1

Continuous functions and convergent sequences

1. Continuous functions

Definition 1.1. Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be two topological spaces and let $f : X \to Y$ be a function.

(a) We say that f is *continuous at* $x \in X$ if for every neighborhood V of $f(x)$ there exists a neighborhood U of x such that $f(U) \subset V$.
(b) We say that f is *continuous* if $f^{-1}(V) \in \mathcal{T}_X$ for every $V \in \mathcal{T}_Y$.

We shall use the term *map* as synonymous with "continuous function."

This definition of continuity, both locally at a point x as well as globally on the entire space X, is motivated by our work on continuity of functions on Euclidean spaces from chapter **??**. Specifically, the reader will recognize that this definition mirrors the results of theorem **??** which considered the case of $X = \mathbb{R}^n$ and $Y = \mathbb{R}^m$.

In the Euclidean case, a function $f : \mathbb{R}^n \to \mathbb{R}^m$ is defined to be continuous globally if it is continuous at every point $x \in \mathbb{R}^n$. Definition 1.1 is set up slightly differently in that global continuity (part (b) of definition 1.1) is not defined as local continuity (part (a) of definition 1.1) at every point $x \in X$. Nevertheless, the relation of global to local continuity of functions between topological spaces remains the same as in the Euclidean space, as explained by the next theorem.

Theorem 1.2. *A function $f : X \to Y$ between two topological spaces is continuous if and only if it is continuous at every point $x \in X$.*

Proof. $\boxed{\Longrightarrow}$ Suppose that $f : X \to Y$ is continuous globally and let x be a point in X. Pick an arbitrary neighborhood V of $f(x)$ in Y, then $U = f^{-1}(V)$ is a neighborhood of x in X with $f(U) \subset V$.

$\boxed{\Longleftarrow}$ Suppose that $f : X \to Y$ is continuous at every point $x \in X$, let V be an open subset of Y and set $U = f^{-1}(V)$. We'd like to show that U is open. For that purpose, let $x \in U$ be an arbitrary point (if U is the empty set then it is automatically open) and note that V is a neighborhood of $f(x)$. By continuity of f at x, there must exist a neighborhood U_x of x with $f(U_x) \subset V$. This latter relation shows that $U_x \subset U$ and consequently we obtain $U = \cup_{x \in U} U_x$. Since U is a union of open sets, it must be an open set. $\square$

Before exploring the concept of continuity into greater depth, we consider some examples.

EXAMPLE 1.3. Let $(X, \mathcal{T}_X) = (\mathbb{R}, \mathcal{T}_{cc})$, let $(Y, \mathcal{T}_Y) = (\mathbb{R}, \mathcal{T}_{fc})$ with $p = 0$ and let $f : X \to Y$ be the function $f(x) = x^2$. Let $U = \mathbb{R} - \{x_1, ..., x_k\} \in \mathcal{T}_{fc}$ be any open set so that $f^{-1}(U) = \mathbb{R} - \{-x_1, x_1, ..., -x_k, x_k\} \in \mathcal{T}_{cc}$. Thus f is continuous.

EXAMPLE 1.4. Let $(X, \mathcal{T}_X) = (\mathbb{R}, \mathcal{T}_{fc})$, let $(Y, \mathcal{T}_Y) = (\mathbb{R}, \mathcal{T}_p)$ with $p = 0$ and let $f : X \to Y$ be again the function $f(x) = x^2$. Then $\{0, 1\} \in \mathcal{T}_p$ but $f^{-1}(\{0, 1\}) = \{-1, 0, 1\} \notin \mathcal{T}_{fc}$ so that f is not continuous.

EXAMPLE 1.5. Let X be a non-empty set and let $\mathcal{P}_1$ and $\mathcal{P}_2$ be two partitions on X and let $\mathcal{T}_1$ and $\mathcal{T}_2$ be the two associated partition topologies on X. Let $f : X \to X$ be the identity function $f(x) = x$ whose domain is equipped with $\mathcal{T}_1$ and codomain with $\mathcal{T}_2$. Then f is continuous if and only if every element in $\mathcal{P}_2$ is a union of elements from $\mathcal{P}_1$.

EXAMPLE 1.6. The constant function $f : X \to Y$, given by $f(x) = p \in Y$ for all $x \in X$, is always continuous since $f^{-1}(U)$ is either the empty set if $p \notin U$ or all of X if $p \in U$.

EXAMPLE 1.7. Let X be equipped with the discrete topology, then any function $f : X \to Y$ is continuous. Conversely, if X is given the indiscrete topology, then a function $f : X \to Y$ to a Hausdorff space Y is continuous if and only if it is constant. For if f were not constant then we could find two points $a, b \in X$ with $f(a) \neq (b)$. The Hausdorff property would guarantee the existence of two open and disjoint sets $U, V \subset Y$ with $f(a) \in U$ and $f(b) \in V$. But then $f^{-1}(U)$ wouldn't be the empty set (since it'd contain a) nor all of X (since it couldn't contain b).

EXAMPLE 1.8. Let (X, d_X) and (Y, d_Y) be two metric spaces and let $\mathcal{T}_X$ and $\mathcal{T}_Y$ be the associated metric topologies. Then a function $f : X \to Y$ is continuous at $x \in X$ if and only if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$x' \in X \text{ and } d_X(x, x') < \delta \qquad \Longrightarrow \qquad d_Y(f(x'), f(x)) < \varepsilon$$

Thus continuous functions between metric spaces satisfy the familiar rule for continuity at a point from analysis. We leave the verification of this as an exercise (exercise **??**), it follows along the lines of the proof of theorem **??**.

The next theorem provides a number of alternative definitions of continuity of functions. Part (a) is a generalization of theorem **??** from the Euclidean case to general topological spaces.

THEOREM 1.9. *Let $f : (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y)$ be a function between two topological spaces. Then f is continuous if and only if any of the mutually equivalent conditions below is met:*

(a) *$f^{-1}(B)$ is a closed subset of X for any closed subset B of Y.*
(b) *For all subsets $B \subset Y$, one gets $\overline{f^{-1}(B)} \subset f^{-1}(\bar{B})$.*
(c) *For all subsets $A \subset X$ one gets $f(\bar{A}) \subset \overline{f(A)}$.*
(d) *For all subsets $B \subset Y$ one gets $f^{-1}(Int(B)) \subset Int(f^{-1}(B))$.*

(e) *Given a basis* $\mathcal{B} = \{V_i \subset Y \mid i \in \mathcal{I}\}$ *for* $\mathcal{T}_Y$, $f^{-1}(V_i)$ *is open for every* $i \in \mathcal{I}$.

PROOF. We will show that properties (a) and (e) are each equivalent to f being continuous. We will then prove the implications (a)$\Longrightarrow$(b)$\Longrightarrow$(c)$\Longrightarrow$(a). Showing that (d) is equivalent to the continuity of f is left as an exercise (exercise 3.1).

(a) Suppose that f is continuous and let $B \subset Y$ be a closed set. Then $f^{-1}(B) = X - f^{-1}(Y - B)$ is also closed since $Y - B$ is open and continuity of f forces $f^{-1}(Y - B)$ to be open also.

Conversely, suppose that f has property (a) and let $V \subset Y$ be any open set. Then $f^{-1}(V) = X - f^{-1}(Y - V)$ is open since $Y - V$ and $f^{-1}(Y - V)$ are both closed, the latter by property (a).

(a)$\Longrightarrow$(b) Let B be any subset of Y. Since $B \subset \bar{B}$, we obtain $f^{-1}(B) \subset f^{-1}(\bar{B})$. By property (a) of f, the set $f^{-1}(\bar{B})$ is closed but since the set $\overline{f^{-1}(B)}$ is the smallest closed set containing $f^{-1}(B)$ (see lemma **??**), the inclusion $\overline{f^{-1}(B)} \subset f^{-1}(\bar{B})$ is immediate.

(b)$\Longrightarrow$(c) Take $A \subset X$ to be any subset of X and apply property (b) of f to $B = f(A)$ to obtain $\overline{f^{-1}(f(A))} \subset f^{-1}(\overline{f(A)})$. Since $A \subset f^{-1}(f(A))$ we also get $\bar{A} \subset \overline{f^{-1}(f(A))}$. Applying f to the inclusion $\bar{A} \subset f^{-1}(\overline{f(A)})$ yields the desired result.

(c)$\Longrightarrow$(a) Let B be a closed subset of Y and set $A = f^{-1}(B)$. Pick a point $x \in \bar{A}$. Then, according to property (b) of f, we must have $f(x) \in \overline{f(A)} = \bar{B} = B$ showing that $x \in A$. This implies that $\bar{A} = A$ and thus that A is closed.

(d) The necessity of property (d) for a continuous function is obvious. Suppose then that f possesses property (d) and let $V \subset Y$ be an open set. Let $\mathcal{J} \subset \mathcal{I}$ be such that $V = \cup_{j \in \mathcal{J}} V_j$. Then $f^{-1}(V) = f^{-1}(\cup_{j\in\mathcal{J}} V_j) = \cup_{j\in\mathcal{J}} f^{-1}(V_j)$ showing that $f^{-1}(V)$ is a union of open sets and therefore open. $\square$

The reader may have noticed that of the various characterizations of continuity in theorem 1.9, there are two conditions (conditions (b) and (c)) involving the closure of sets but only one condition (condition (d)) involving the interior. Condition (b) involves taking preimages under f and condition (c) requires one to take images under f. Given that condition (d) also involves preimages, one may suspect that perhaps there is also a characterization of continuity of f in terms of taking images of f and interiors of sets. That this is not so is shown by the next two examples.

EXAMPLE 1.10. Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ given by $f(x, y) = x$ and with each of $\mathbb{R}^2$ and $\mathbb{R}$ equipped with the Euclidean topology. Then f is clearly continuous, but, taking $A = \{(x, 0) \in \mathbb{R}^2 \mid x \in \mathbb{R}\}$, we find that

$$Int(f(A) = Int(\mathbb{R}) = \mathbb{R} \quad \text{while} \quad f(Int(A)) = f(\emptyset) = \emptyset$$

Thus the inclusion $Int(f(A) \subset f(Int(A))$ fails in general for continuous functions.

EXAMPLE 1.11. Consider the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = 0$ and assume again that both copies of $\mathbb{R}$ come with the Euclidean topology. Pick $A = \mathbb{R}$, then

$$f(Int(A)) = f(\mathbb{R}) = \{0\} \quad \text{while} \quad Int(f(A)) = Int(\{0\}) = \emptyset$$

Thus the inclusion $f(Int(A)) \subset Int(f(A))$ also fails in general for continuous functions (see however part (a) of proposition 1.17 below).

We next single out some simple functions that are always continuous. The reader will no doubt recognize familiar properties of continuous functions from the Euclidean case.

PROPOSITION 1.12. *Let $f : X \to Y$ and $g : Y \to Z$ be continuous functions between topological spaces and let $A \subset X$ be any subspace of X. Then*

(a) *The inclusion function $\iota : A \to X$ is continuous. In particular, the identity function id: $X \to X$ is always continuous.*
(b) *The composition function $g \circ f : X \to Z$ is continuous.*
(c) *The restriction function $f|_A : A \to Y$ is continuous.*
(d) *Let $U_i \subset X$, $i \in \mathcal{I}$, be a collection of open subsets of X such that $X = \cup_{i\in\mathcal{I}} U_i$ and let $h : X \to Y$ be a function with $h|_{U_i} : U_i \to Y$ continuous for each $i \in \mathcal{I}$. Then h is continuous.*

PROOF. (a) For an open subset $U \subset X$, the preimage $\iota^{-1}(U)$ equals $U \cap A$ and is therefore open on A with respect to its relative topology.

(b) Let W be an open subset of Z. Then $V = g^{-1}(W)$ is an open subset of Y and thus $U = f^{-1}(V)$ must be open in X. Since $U = f^{-1}(g^{-1}(W) = (g \circ f)^{-1}(W)$, we're done.

(c) This follows from parts (a) and (b) since $f|_A = f \circ \iota$ where $\iota : A \to X$ is the inclusion map.

(d) Set $h_i = h|_{U_i}$ and let $V \subset Y$ be an open set. Then $h^{-1}(V) = \cup_{i\in\mathcal{I}} h_i^{-1}(V)$ and, since each $h_i^{-1}(V)$ must be open in U_i (in its relative topology), there must be open subsets $W_i \subset X$ with $h_i^{-1}(V) = U_i \cap W_i$. As both U_i and W_i are open in X then so is $U_i \cap W_i$ showing that $h^{-1}(V)$ is a union of open sets and therefore open. □

DEFINITION 1.13. A function $f : X \to Y$ between topological spaces is called a *homeomorphism* if f is a continuous bijection with a continuous inverse function $f^{-1} : Y \to X$. We say that two topological spaces X and Y are *homeomorphic*, and write $X \cong Y$, if there exists at least one homeomorphism $f : X \to Y$.

A function $f : X \to Y$ is called a *local homeomorphism* if every point $x \in X$ has a neighborhood U such that $f|_U : U \to f(U)$ is a homeomorphism, where U and $f(U)$ come equipped with their relative topologies inherited from X and Y respectively. Two spaces are called *locally homeomorphic* if there exists at least one local homeomorphism between them.

Talk about how homeomorphic spaces are really one and the same from an abstract point of view.

REMARK 1.14. The relation of "being homeomorphic to" between topological spaces is an equivalence relation. By this we mean that it satisfies the following three properties:

1. Reflexivity: $X \cong X$.

2. Symmetry: If $X \cong Y$ then also $Y \cong X$.
3. Transitivity: If $X \cong Y$ and $Y \cong Z$ then $X \cong Z$.

Part 1 is facilitated by the identity map id: $X \to X$ (see proposition 1.12, part 3). For part two, if $f : X \to Y$ is a homeomorphism, then $f^{-1} : Y \to X$ is also a homeomorphism. Finally, for part 3, if $f : X \to Y$ and $g : Y \to Z$ are homeomorphisms, then $g \circ f : X \to Z$ is also a homeomorphism (see proposition 1.12 part 2).

In a similar vein one can also prove that the relation of "being locally homeomorphic"is an equivalence relations among topological spaces (see exercise **??**).

Example 1.15.

Definition 1.16. Let $f : X \to Y$ be a map between topological spaces. We say that f *is open* if $f(U)$ is an open subset of Y whenever U is an open subset of X. Similarly, we say that f *is closed* if $f(A)$ is closed in Y for every choice of a closed subset $A \subset X$.

Note that, by definition, every homeomorphism is an open map. Indeed the condition for a bijection f to be open is the same as the condition as for f^{-1} to be continuous. Similarly, using theorem 1.9, one finds that a homeomorphism is also a closed map. Perhaps more surprisingly, local homeomorphisms are also open maps:

Proposition 1.17. *Let $f : X \to Y$ be a map between topological spaces.*

(a) *f is an open map if and only if $f(Int(A)) \subset Int(f(A))$.*

b) *f is a closed map if and only if $\overline{f(A)} = f(\bar{A})$ for every subset $A \subset X$.*

(c) *If f is a local homeomorphism then f is an open map.*

Proof. (a) Suppose first that f is open and let A be any subset of X. Since $Int(A) \subset A$ we find that $f(Int(A)) \subset f(A)$. Since f is open, the set $f(Int(A))$ is an open subset of $f(A)$. But $Int(f(A))$ is the largest open subset of $f(A)$ forcing the inclusion $f(Int(A)) \subset Int(f(A))$.

Conversely, suppose that $f(Int(A)) \subset Int(f(A))$ holds for all $A \subset X$. Pick an open subset $U \subset X$, then $f(Int(U)) = f(U) \subset Int(f(U))$. On the other hand, the inclusion $Int(f(U)) \subset f(U)$ is trivially true showing that $f(U) = Int(f(U))$, thus $f(U)$ is open.

(b) Assume that f is closed and that $A \subset X$ is any set. Then $A \subset \bar{A}$ implies that $f(A) \subset f(\bar{A})$. Since $f(\bar{A})$ is closed and contains $f(A)$, the inclusion $\overline{f(A)} \subset f(\bar{A})$ follows since the closure of $f(A)$ is the smallest closed set containing $f(A)$. The converse inclusion $f(\bar{A}) \subset \overline{f(A)}$ follows from theorem 1.9, part (c).

On the other hand, suppose that $\overline{f(A)} = f(\bar{A})$ for all subsets A of X, we'd like to show that f is closed. Thus, let $B \subset X$ be any closed set, then $f(B) = f(\bar{B}) = \overline{f(B)}$, the latter set is of course closed.

(c) Assume that f is a local homeomorphism. To show that f is an open map, let $U \subset X$ and set $V = f(U)$. We'd like to show that V is an open subset of Y. Towards this goal, pick a point $x \in U$. Then there exists a neighborhood U_x of x such that $f|_{U_x} : U_x \to V_x$, with $V_x = f(U_x)$, is a homeomorphism. Without loss of generality we can assume that $U_x \subset U$ for if not, we could simply replace U_x by $U_x \cap U$. But then

$V_x \subset V$ and therefore $V = \cup_{x \in U} V_x$. Being a union of open sets, V is forced to be open itself. □

The next example shows that a local homeomorphism does not have to be a closed map.

EXAMPLE 1.18. Let $X = \langle 0, 1\rangle \subset \mathbb{R}$ and $Y = \{(x, y) \in \mathbb{R}^2 \,|\, x^2 + y^2 = 1\} \subset \mathbb{R}^2$, each equipped with the relative Euclidean topology. Consider the map $f : X \to Y$ defined by $f(t) = (\cos(2\pi t), \sin(2\pi t))$. This is a local homeomorphism but $f(X) = Y - \{(1, 0)\}$ and X is closed while $Y - \{(1, 0)\}$ is not.

We will encounter open and closed maps again in subsequent chapters. Open maps will play an important role in the study of quotient spaces (chapter **??**). Chapter **??** will give a nice criterion for a map to be closed in terms of compactness of the domain of the map (theorem **??**).

2. Convergent sequences

In this section we examine convergent sequences in topological spaces, first in the their own right and then with regards to their relation to continuous function. Familiar properties of sequences from Euclidean spaces no longer hold in this more general setting, perhaps most striking being the non-uniqueness phenomena for the limit of a convergent sequence. A characterization of continuity in terms of sequences is given in theorem 2.12.

DEFINITION 2.1. Let $(X, \mathcal{T})$ be a topological space and $x_k \in X$ a sequence. We say that the sequence x_k *converges to* $x \in X$, and write $\lim_{k\to\infty} x_k = x$ or just $\lim x_k = x$, if for every neighborhood U of x there exists a $k_0 \in \mathbb{N}$ such that for all $k \geq k_0$ we obtain $x_k \in U$.

Note that if one takes $(X, \mathcal{T}_X) = (\mathbb{R}^n, \mathcal{T}_{Eu})$ then this definition agrees with the familiar definition of continuity in Euclidean spaces, compare to theorem **??**.

EXAMPLE 2.2. Let $X = \mathbb{R}$ be equipped with the partition topology $\mathcal{T}_{\mathcal{P}}$ associated to the partition

$$\mathcal{P} = \{[4a - 2, 4a + 2\rangle \,|\, a \in \mathbb{Z}\}$$

Then the sequence $x_n = (-1)^n$ converges to any point $x \in [-2, 2\rangle$ since this is the smallest non-empty open set containing both 1 and -1.

EXAMPLE 2.3. If X is equipped with the indiscrete topology, then any sequence in X is convergent and its limit is any point in X. Conversely, if X is given the discrete topology, then a sequence x_k is convergent to $x \in X$ if and only if $x_k = x$ for all sufficiently large k.

EXAMPLE 2.4. Let $X = \mathbb{R}$ be given the finite complement topology $\mathcal{T}_{fc}$ and consider the sequence $x_k = k$. Then $lim x_k = x$ for any $x \in \mathbb{R}$ since every neighborhood of x contains all but finitely many elements of $\{x_1, x_2, x_3, ...\}$. If we give $\mathbb{R}$ the countable complement topology $\mathcal{T}_{cc}$ instead, then $\lim x_k = x$ if and only if $x_k = x$ for all k sufficiently large (exercise **??**).

EXAMPLE 2.5. **5.** On $X = \mathbb{R}$ consider the particular point topology $\mathcal{T}_p$ with $p = 0$. Then the only sequences converging to $p = 0$ are the sequences which are eventually constant (and equal to zero) since $\{0\}$ is a neighborhood of 0. The sequences converging to $\neq 1 \in \mathbb{R}$ are those that lie in the set $\{0, x\}$ for all sufficiently large indices. to $x_0 = 1$!

As the examples above show, in general topological spaces limits of sequences may not be unique. They are unique tough if X happens to have the Hausdorff property.

DEFINITION 2.6. A topological space X is called *Hausdorff* if every two points have disjoint neighborhoods. Said differently, we require that for each pair of points $a, b \in X$, $a \neq b$, there exist open sets $U_a, U_b \subseteq X$ with $a \in U_a$, $b \in U_b$ and $U_a \cap U_b = \emptyset$.

THEOREM 2.7. *Let X be a topological space.*

(a) *If X Hausdorff space and $x_k \in X$ is a convergent sequence, then the limit* $\lim x_k$ *is unique.*
(b) *If X is fist countable and has the property that every convergent sequence $x_k \in X$ has a unique limit, then X is Hausdorff.*

PROOF. (a) Suppose that there are two (or more) limits for x_k, say a and b. Since X is Hausdorff, we can find disjoint neighborhoods U_a and U_b of a and b respectively. Let $k_a \in \mathbb{N}$ be such that $x_k \in U_a$ for all $k \geq k_a$ and $k_b \in \mathbb{N}$ have the property that $x_k \in U_b$ for all $k \geq k_b$. Then for all $k \geq \max\{k_a, k_b\}$ we have that $x_k \in U_a \cap U_b$, a contradiction since $U_a \cap U_b = \emptyset$.

(b) Given two arbitrary points $a, b \in X$ with $a \neq b$, we need to find two disjoint open sets of which one contains a and the other contains b. Let $\mathcal{B}_a = \{U_i^a \subset X \mid i \in \mathbb{N}\}$ and $\mathcal{B}_b = \{U_i^b \subset X \mid i \in \mathbb{N}\}$ be countable neighborhood bases at a and b respectively. We define new open sets V_i^a and V_i^b as

$$V_k^a = U_1^a \cap U_2^a \cap \ldots \cap U_k^a \qquad \text{and} \qquad V_k^b = U_1^b \cap U_2^b \cap \ldots \cap U_k^b$$

These sets are open since they are finite intersections of open sets. Furthermore, notice that $V_j^a \subset U_i^a$ and $V_j^b \subset U_i^b$ for every $j \geq i$. Clearly, each V_i^a is a neighborhood of a and each V_i^b is a neighborhood of b.

If $V_k^a \cap V_k^b = \emptyset$ for some k, we are done. So suppose instead that $V_k^a \cap V_k^b \neq \emptyset$ for all $k \in \mathbb{N}$. Let $x_k \in V_k^a \cap V_k^b$ be an arbitrary point. This yields a sequence in X which we claim converges to a. To see this, let U be any neighborhood of a and find an $U_{k_a}^a \in \mathcal{B}_a$, $k_a \in \mathbb{N}$, such that $U_{k_a}^a \subset U$. Then $V_k^a \subseteq U$ for every $k \geq k_a$, in particular $x_k \in U$ for every $k \geq k_a$, showing that $\lim x_k = a$. Repeating this same argument for b shows also that $\lim x_k = b$. This is a contradiction since by assumption all convergent sequences in X have a unique limit. We are thus forced to conclude that there is some $k \in \mathbb{N}$ for which $V_k^a \cap V_k^b = \emptyset$, giving us disjoint neighborhoods of a and b, as needed. □

The first countability condition from part (b) of theorem 2.7 is necessary and cannot be weakened as the next example shows.

EXAMPLE 2.8. Consider $X = \mathbb{R}$ equipped with the countable complement topology. We claim that every convergent sequence $x_k \in X$ has a unique limit. Suppose not, that is suppose that $\lim x_k = a$ and $\lim x_k = b$ with $a \neq b$. Let U_a be the open set

$$U_a = \mathbb{R} - \{x_i \,|\, x_i \neq a\}$$

Clearly $a \in U_a$ and so there must be some $n_a \in \mathbb{N}$ such that $x_n \in U_a$ for all $n \geq n_a$. But then $x_n = a$ for all $n \geq n_a$ since $x_n \in U_a \cap \{x_i \,|\, i \in \mathbb{N}\} = \{a\}$. A similar argument shows that for some $n_b \in \mathbb{N}$ all $x_n = b$ for $n \geq n_b$. But then $x_n = a$ and $x_n = b$ for $n \geq \max\{n_a, n_b\}$ which is impossible since $a \neq b$. On the other hand, X is not Hausdorff since every two non-empty open sets have nontrivial intersection.

Recall that we saw that a subset $A \subseteq \mathbb{R}^n$ (with the Euclidean topology) is closed if and only if it contains the limits of all its convergent sequences (this was our original definition of a closed subset of $\mathbb{R}^n$, see definition **??**). This characterization of closed sets is only partly true in general topological spaces.

THEOREM 2.9. *Let X be a topological space and A a subset of X.*

(a) *If A is a closed then it contains the limits of all its convergent sequences.*
(b) *If X is first countable and A contains the limits of all of its convergent sequences, then A is closed.*

PROOF. (a) Let $x_k \in A$ be a sequence with limit x. If $x \notin A$ then $x \in X - A$ which is an open set. By convergence of x_k there must be some $k_0 \in \mathbb{N}$ such that $x_k \in X - A$ for all $k \geq k_0$. This is impossible since $x_k \in A$ and $A \cap (X - A) = \emptyset$.

(b) We will show that A is closed by exhibiting that $A = \bar{A}$. Suppose this were not true. Then $\bar{A} - A$ would be nonempty. Let $x \in \bar{A} - A$ and let $\mathcal{B}_x = \{U_i \subseteq X \,|\, i = 1, 2, 3, ...\}$ be a countable basis at x and define V_i as

$$V_j = U_1 \cap U_2 \cap ... \cap U_j$$

Note that the sets V_i are open, that the inclusion $V_j \subset U_i$ holds for all $j \geq i$ and that each V_j contains x. We claim next that each set $V_i \cap A$ must be non-empty. For if not then $\bar{A} - V_j$ would be a closed set smaller than $\bar{A}$ and containing A, a contradiction. Thus we can pick an element $x_k \in V_k \cap A$. But now x_k must converge to x since if V is any neighborhood of x then there must be an index k_0 such that $x \in U_{k_0} \subset V$ and thus $x_k \in V$ for all $k \geq k_0$. Since $x_k \in A$ and $\lim x_k = x$ we conclude that $x \in A$. Therefor $\bar{A} = A$. □

That the first countability condition from part (b) of the preceding theorem cannot be dropped, is illustrated by the next example.

EXAMPLE 2.10. Let $X = \mathbb{R}$ be equipped with the countable complement topology and let $A \subseteq X$ be the set $A = X - \{0\}$. Notice that A is not closed (since $X - A = \{0\}$ is not open). But A contains the limits of all of its convergent subsequences. To see this we only need to show that no sequence $x_k \in A$ can converge to 0. This is easy to see since the set $U = X - \{x_1, x_2, ...\}$ is an open set which contains zero but no element of the sequence x_n. Thus A contains the limits of all of its convergent sequences. Note that this, in conjunction with theorem 2.9, show that $(\mathbb{R}, \mathcal{T}_{cc})$ cannot be first countable and hence neither second countable.

Parts (b) from both theorem 2.7 and theorem 2.9 show that if we assume the topological space X in question to be first countable, then sequences in X behave in close analogy with their Euclidean counterparts. The next theorem shows that this kinship extends even further.

THEOREM 2.11. *Let X be a topological space and let $x \in X$ be a given point. Suppose the $\{x\}$ is not an open set and that x has a countable neighborhood basis. Then there exists a sequence $x_k \in X - \{x\}$ converging to x.*

PROOF. Suppose that $\mathcal{B}_x = \{U_1, U_2, U_3, ...\}$ is a countable neighborhood basis at $x \in X$. As in the proofs of theorems 2.7 and 2.9, we define new set V_j as

$$V_k = U_1 \cap U_2 \cap ... \cap U_k$$

Since $\{x\}$ is not an open set by assumption, we cannot have $V_k = \{x\}$ for any value of $k \in \mathbb{N}$. Thus, we can always pick an element $x_k \in V_k - \{x\}$ and thus form a sequence. It is easy to prove that $\lim x_k = x$ (this was already shown in the both the proof of theorem 2.7 and in the proof of theorem 2.9). □

We conclude this section by examining the relation between continuous functions and convergent sequences. As the reader may guess by now, with the assumption of first countability, the relation between the two is just as in Euclidean space (see theorem **??**).

THEOREM 2.12. *Let $f : X \to Y$ be a function between two topological spaces.*

(a) *If f is continuous at $x \in X$ and if $x_k \in X$ is a convergent sequence with $\lim x_k = x$, then $f(x_k) \in Y$ is also a convergent sequence with $\lim f(x_k) = f(x)$.*
(b) *If X is first countable and f has the property that $lim f(x_k) = f(x)$ for all sequences $x_k \in X$ converging to x, then f is continuous at x.*

PROOF. (a) We will first show that property (a) of the theorem is equivalence to continuity of f. Let $V \subseteq Y$ be a neighborhood of $f(x)$. Since f is continuous, the set $U = f^{-1}(V) \subseteq X$ is a neighborhood of x. Since $\lim x_k = x$ there is some $k_0 \in \mathbb{N}$ such that $x_k \in U$ for all $k \geq k_0$. But then $f(x_k) \in V$ for all $k \geq k_0$ also, showing that $\lim f(x_k) = f(x)$.

(b) Let V be a neighborhood of $f(x)$ and set $U = f^{-1}(V)$. We seek to show that U contains a neighborhood of x. Suppose that this were not so. In that case, let $\mathcal{B}_x = \{V_1, V_2, V_3, ...\}$ be a neighborhood basis around x and pick $x_k \in (X - U) \cap V_k$ arbitrarily. Then x_k converges to x and thus by assumption $f(x_k)$ must converge to $f(x)$. This however is impossible since V is open and $f(x) \in V$ while $f(x_k) \notin V$. Consequently, U must contain a neighborhood of x and hence f is continuous at x. □

3. Exercises

3.1. Show that a function $f : X \to Y$ is continuous if and only if $f^{-1}(Int(B)) \subset Int(f^{-1}(B))$ for every subset $B \subset Y$.

3.2. For $\lambda \in \mathbb{R}$ let $f_\lambda : \mathbb{R} \to \mathbb{R}$ be the function $f_\lambda(x) = x + \lambda$. If both the domain and codomain of f_λ are equipped with the finite complement topology, for which $\lambda \in \mathbb{R}$ is f_λ continuous?