PRODUCT FORMULAE FOR OZSVÁTH-SZABÓ 4-MANIFOLD INVARIANTS

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November 6, 2005

Abstract. We give formulae for the Ozsváth-Szabó invariants of 4-manifolds $X$ obtained by fiber sum of two manifolds $M_1, M_2$ along surfaces $\Sigma_1, \Sigma_2$ having trivial normal bundle and genus $g \geq 1$. The formulae follow from a general result on the Ozsváth-Szabó invariants of the result of gluing two 4-manifolds along a common boundary, which is phrased in terms of relative invariants of the pieces. The fiber sum formula follows from this theorem along with previous results of the authors on the Heegaard Floer homology of manifolds of the form $\Sigma \times S^1$. The product formulae lead quickly to calculations of the Ozsváth-Szabó invariants of various 4-manifolds; in all cases the results are in accord with the conjectured equivalence between Ozsváth-Szabó and Seiberg-Witten invariants.

1. Introduction

The theory of smooth 4-manifolds abounds with exotica. At the time of writing, there is no example of a smoothable topological 4-manifold whose smooth structures have been classified—indeed, no 4-manifold is known to support only finitely many smooth structures, and in virtually every case a 4-manifold that admits more than one smooth structure is known to admit infinitely many such structures. The construction of these examples, due to authors too numerous to list (see [12] for a brief survey), makes use of various techniques but in many cases involves the “fiber sum” operation in one guise or another. In this paper, we give formulae that essentially determine the behavior of the Ozsváth-Szabó 4-manifold invariants under fiber sum.

The Ozsváth-Szabó invariants [8, 9] are defined using a “TQFT” construction, meaning that they are built from invariants of 3-dimensional manifolds (the Heegaard Floer homology groups) and cobordisms between such manifolds. To a closed oriented 4-manifold $M$ with $b^+(M) \geq 2$, with a spin$^c$ structure $\mathfrak{s}$, Ozsváth and Szabó associate a linear function

$$\Phi_{M,\mathfrak{s}} : \mathbb{A}(M) \to \mathbb{Z}/\pm 1,$$

where $\mathbb{A}(M)$ is the algebra $\Lambda^*(H_1(M;\mathbb{Z})/\text{torsion}) \otimes \mathbb{Z}[[U]]$, graded such that elements of $H_1(M)$ have degree 1 and $U$ has degree 2. This invariant has the property that $\Phi_{M,\mathfrak{s}}$ is nonzero for at most finitely many spin$^c$ structures $\mathfrak{s}$, and furthermore can be nonzero only on homogeneous elements of $\mathbb{A}(M)$ having degree

$$d(\mathfrak{s}) = \frac{1}{4}(c_2^+(\mathfrak{s}) - 3\sigma(M) - 2e(M)),$$

where $\sigma$ denotes the signature of the intersection form on $M$ and $e$ is the Euler characteristic. Ozsváth and Szabó conjecture [8] that $\Phi_{M,\mathfrak{s}}$ is identical with the Seiberg-Witten invariant, which has been a key tool in distinguishing the smooth structures on fixed topological manifolds alluded to above.
We should remark that there is a sign ambiguity in the definition of \( \Phi_{M,s} \), so that the results to follow should be interpreted as true up to an overall sign. Furthermore the invariants can be extended to 4-manifolds having \( b^+(M) = 1 \), but in this case depend on an additional choice of a class \( v \in H_2(M; \mathbb{Q}) \) with \( v.v = 0 \) (see [11] and below), which we omit from the notation.

The fiber sum of two smooth 4-manifolds is defined as follows. Let \( M_1 \) and \( M_2 \) be closed oriented 4-manifolds, and suppose \( f_i : \Sigma \to M_i \), \( i = 1, 2 \), are smooth embeddings of a closed oriented surface \( \Sigma \) in \( M_i \). We assume throughout this paper that the genus of \( \Sigma \) is at least 1 and that the embeddings \( f_i \) have trivial normal bundles. In this case, the images \( \Sigma_i = f_i(\Sigma) \) have neighborhoods \( N(\Sigma_i) \) diffeomorphic to \( \Sigma \times D^2 \). Choose an orientation-reversing diffeomorphism \( \phi : \partial N(\Sigma_1) \to \partial N(\Sigma_2) \) that lifts \( f_2 \circ f_1^{-1} : \Sigma_1 \to \Sigma_2 \), and define the fiber sum \( X = M_1 \#_\Sigma M_2 \) by

\[
X = (M_1 \setminus N(\Sigma_1)) \cup_\phi (M_2 \setminus N(\Sigma_2)).
\]

In general, the manifold \( M \) can depend on the choice of lift \( \phi \). We assume henceforth that the homology classes \( [\Sigma_1] \) and \( [\Sigma_2] \) are non-torsion elements of \( H_2(M_i; \mathbb{Z}) \).

To state the results, it is convenient to express the Ozsváth-Szabó invariant in terms of the group ring \( \mathbb{Z}[H^2(M; \mathbb{Z})] \). That is to say, we write

\[
OS_M = \sum_{s \in \text{Spin}^c(M)} \Phi_{M,s} e^{c_1(s)},
\]

where \( e^{c_1(s)} \) is the formal variable in the group ring corresponding to the first Chern class of the spin\(^c\) structure \( s \) (note that \( c_1(s) = c_1(s') \) for distinct spin\(^c\) structures \( s \) and \( s' \) iff \( s - s' \) is of order 2 in \( H^2(M; \mathbb{Z}) \), so the above formulation may lose some information if 2-torsion is present). The coefficients of the above expression are functions on \( \mathbb{A}(M) \), so that \( OS_M \) is an element of \( \mathbb{Z}[H^2(M; \mathbb{Z})] \otimes \mathbb{A}(M)^* \). The value of the invariant on \( \alpha \in \mathbb{A}(M) \) is denoted \( OS_M(\alpha) \in \mathbb{Z}[H^2(M; \mathbb{Z})] \).

The behavior of \( \Phi_{M,s} \) under fiber sum depends on the value of \( \langle c_1(s), [\Sigma] \rangle \) (note that since \( c_1(s) \) is a characteristic class, this value is always even when \( [\Sigma]^2 = 0 \), so we partition \( OS_M \) accordingly: for an embedded surface \( \Sigma \hookrightarrow M \) with trivial normal bundle, let

\[
OS_M^k = \sum_{\langle c_1(s), [\Sigma] \rangle = 2k} \Phi_{M,s} e^{c_1(s)}.
\]

The adjunction inequality for Ozsváth-Szabó invariants implies that \( OS_M^k \equiv 0 \) if \( |k| > g - 1 \).

The topology of fiber sums is complicated in general by the presence of rim tori. A rim torus is a submanifold of the form \( \gamma \times S^1 \subset \Sigma \times S^1 \), where \( \gamma \) is an embedded circle on \( \Sigma \). Such tori are homologically trivial in the fiber summands \( M_i \), but typically essential in \( X = M_1 \#_\Sigma M_2 \). Let \( \mathcal{R} \) denote the subspace of \( H^2(X; \mathbb{Z}) \) spanned by the Poincaré duals of rim tori, and let \( \rho : H^2(X; \mathbb{Z}) \to H^2(X; \mathbb{Z})/\mathcal{R} \) denote the natural projection. If \( b_i \in H^2(M_i; \mathbb{Z}) \), \( i = 1, 2 \), are cohomology classes with the property that \( b_i|_{\partial N(\Sigma_i)} \) agrees with \( b_2|_{\partial N(\Sigma_2)} \) under \( \phi \), then Mayer-Vietoris arguments show that there exists a class \( b \in H^2(X; \mathbb{Z}) \) whose restrictions to \( M_i \setminus N(\Sigma_i) \) agrees with the corresponding restrictions of \( b_i \), and furthermore that \( b \) is determined uniquely up to elements of \( \mathcal{R} \) and multiples of the Poincaré dual of \( \Sigma \). If \( b, b_1 \) and \( b_2 \) satisfy these conditions on their respective restrictions, we say that the three classes...
are compatible with the fiber sum. We can eliminate part of the ambiguity in $b$ given $(b_1, b_2)$ by requiring that
\[(2) \quad b^2 = b_1^2 + b_2^2 + 4|m|,\]
where $m = \langle b_1, [\Sigma_1] \rangle = \langle b_2, [\Sigma_2] \rangle$. With this convention, the pair $(b_1, b_2)$ gives rise to a well-defined element of $H^2(X; \mathbb{Z})/\mathcal{R}$ (see section 6.3 for details).

**Theorem 1.1.** Let $X = M_1 \#_\Sigma M_2$ be obtained by fiber sum along a surface $\Sigma$ of genus $g > 1$ from manifolds $M_1$, $M_2$ satisfying $b^+(M_i) \geq 1$, $i = 1, 2$. Then for any $k$ satisfying $\frac{1}{3}(g-2) < |k| \leq g-1$ and any $\alpha \in \mathbb{A}(X)$ we have
\[(3) \quad \rho \left( OS^k_X(\alpha) \right) = \sum_{\beta \in \mathcal{B}_k} OS^k_{M_1}(\alpha_1 \otimes f_{1*}(\beta)) \cdot OS^k_{M_2}(f_{2*}(\beta^\circ) \otimes \alpha_2),\]
where $\alpha_i \in \mathbb{A}(M_i \setminus N(\Sigma_i))$ are any elements such that $\alpha_1 \otimes \alpha_2$ maps to $\alpha$ under the inclusion-induced homomorphism.

When $b^+ = 1$ for any of the manifolds in the theorem, the invariant is taken to be calculated using the choice $v = [\Sigma]$.

The notation of the theorem requires some explanation. First, the product of group ring elements appearing on the right makes use of the construction outlined above, producing elements of $H^2(X; \mathbb{Z})/\mathcal{R}$ from compatible pairs $(b_1, b_2)$. The set $\mathcal{B}_k$ denotes a basis over $\mathbb{Z}$ for the group $H_*(\text{Sym}^d(\Sigma); \mathbb{Z})$, thought of as a subgroup of $\mathbb{A}(\Sigma)$, where $d = g-1-|k|$. Likewise, $\{\beta^\circ\}$ denotes the dual basis to $\mathcal{B}_k$ under a certain nondegenerate pairing (see section 6.2). The terms $\alpha_1 \otimes f_{1*}(\beta)$ and $f_{2*}(\beta^\circ) \otimes \alpha_2$ are understood to mean the images of those elements in $\mathbb{A}(M_1)$ and $\mathbb{A}(M_2)$, using the inclusion-induced maps.

When $k$ is unconstrained, but still nonzero, we have the following slightly less precise result.

**Theorem 1.2.** Let $X, M_1, M_2$ be as in the previous theorem, and suppose $s \in \text{Spin}^c(X)$, $s_i \in \text{Spin}^c(M_i)$ satisfy the property that
\[s|_{M_i \setminus N(\Sigma_i)} = s_i|_{M_i \setminus N(\Sigma_i)} \quad \text{and} \quad \langle c_1(s_i), [\Sigma_i] \rangle = \langle c_1(s), [\Sigma] \rangle = 2k \]
for $i = 1, 2$, where $0 < |k| \leq g-1$. Then for any $\alpha \in \mathbb{A}(X)$,
\[\sum_{n \in \mathbb{Z}} \Phi_{X,s+n\Sigma^*}(\alpha) t^n = t^{n_0} \sum_{\beta \in \mathcal{B}_k, \quad n_1, n_2 \in \mathbb{Z}} \Phi_{M_1,s_1+n_1\Sigma_1^*}(\alpha \otimes f_{1*}(\beta)) \cdot \Phi_{M_2,s_2+n_2\Sigma_2^*}(f_{2*}(\beta^\circ) \otimes \alpha_2) t^{n_1-n_2}\]
as polynomials in $t$, where $n_0 \in \mathbb{Z}$. Here $\Sigma^*$, $\Sigma_1$, $\Sigma_2$ are the classes Poincaré dual to $\Sigma$, $\Sigma_1$, $\Sigma_2$ in $X$, $M_1$, $M_2$ respectively, while $\Phi_{X,t}(\alpha)$ denotes the sum of all invariants $\Phi_{X,t+h}(\alpha)$ where $h \in \mathcal{R}$.

The issue in deducing Theorem 1.1 from Theorem 1.2 is the current imprecise understanding of the dual basis $\beta^\circ$. Indeed, the results above make use of the fact that the Heegaard Floer homology of $\Sigma \times S^1$ is essentially $H_*(\text{Sym}^d(\Sigma))$, and the duality between $\beta$ and $\beta^\circ$ arises using this identification. In the situation of Theorem 1.1, namely $\frac{1}{3}(g-2) < |k| < g-1$, the duality corresponds to intersection duality in $H_*(\text{Sym}^d(\Sigma); \mathbb{Z})$ and this allows for the precise correspondence between spin$^c$ structures on the three manifolds (specifically (2)).
much difference in the application of the two results since one can determine the factor \( n \) and allows only the statement given in Theorem 1.2. In practice, there is unlikely to be much difference in the application of the two results since one can determine the factor \( n \) and allows only the statement given in Theorem 1.2 using other properties of \( \Phi \). More significantly, a 4-manifold \( X \) is said to have (Ozsváth-Szabó) simple type if any spin\(^c\) structure \( s \) for which \( \Phi_{X,s} \neq 0 \) has \( d(s) = 0 \). We have:

**Corollary 1.3.** If \( M_1 \) and \( M_2 \) have simple type, then the fiber sum \( X = M_1 \#_\Sigma M_2 \) has the property that if \( \Phi_{X,s}^{Rim} \neq 0 \) and \( d(s) \neq 0 \), then \( \langle c_1(s), [\Sigma] \rangle = 0 \). Furthermore,

\[
\Phi_{X,s}^{Rim} = 0 \quad \text{if} \quad 0 < |\langle c_1(s), [\Sigma] \rangle| < g - 1.
\]

Finally,

\[
\rho(O^g_{X^{-1}}(\alpha)) = \begin{cases} 
O^g_{M_1}(1) \cdot O^g_{M_2}(1) & \text{if} \quad \alpha = 1 \\
0 & \text{otherwise.}
\end{cases}
\]

In other words, the fiber sum has simple type (after sum over rim tori) with the possible exception of spin\(^c\) structures whose first Chern class pairs trivially with \([\Sigma]\). At this point the obstruction to extending the vanishing result (4) to this last case is the incomplete understanding of the module structure of the Heegaard Floer homology of \( \Sigma \times S^1 \) in the spin\(^c\) structure with trivial first Chern class.

We should remark that Taubes [13] has shown that symplectic 4-manifolds with \( b^+ \geq 2 \) have Seiberg-Witten simple type. It seems safe, therefore, to make the following:

**Conjecture 1.4.** If \( X \) is a symplectic 4-manifold with \( b^+(X) \geq 2 \) then \( X \) has Ozsváth-Szabó simple type.

Leaving this issue for now, we turn to the case of a fiber sum along a torus, where the product formula is slightly different. In this situation, it need not be the case that the homology classes \([\Sigma_1]\) and \([\Sigma_2]\) are identified in the fiber sum. For simplicity, we state the result under the assumption that they are indeed identified.

**Theorem 1.5.** Let \( X = M_1 \#_{\Sigma} M_2 \) be obtained by fiber sum along a surface \( \Sigma \) of genus \( g = 1 \) from manifolds \( M_1, M_2 \) satisfying \( b^+(M_i) \geq 1, \ i = 1, 2 \). Assume furthermore that the map \( H_1(\partial N(\Sigma_i); \mathbb{Z}) \to H_1(M_i \setminus N(\Sigma_i); \mathbb{Z}) \) is trivial for \( i = 1, 2 \), and that \([\Sigma_1] = [\Sigma_2]\) in \( X \). Let \( \tilde{T} \) denote the Poincaré dual of the class in \( H_2(X; \mathbb{Z}) \) induced by \( [\Sigma_1] \), and write \( T \) for the image of \( \tilde{T} \) in \( H^2(X; \mathbb{Z})/\mathcal{R} \). Then for any \( \alpha \in \mathcal{A}(X) \) we have

\[
\rho(OS_X(\alpha)) = (T - T^{-1})^2 OS_{M_1}(\alpha_1) \cdot OS_{M_2}(\alpha_2)
\]

where \( \alpha_1 \otimes \alpha_2 \in \mathcal{A}(M_1) \otimes \mathcal{A}(M_2) \) maps to \( \alpha \) as before.

Here the product between \( OS_{M_1} \) and \( OS_{M_2} \) uses the construction from previously, while multiplication with \( T \) takes place in the group ring of \( H^2(X; \mathbb{Z})/\mathcal{R} \). Again, the invariant for any manifold with \( b^+ = 1 \) in the theorem is taken to be calculated with respect to \( v = [\Sigma_1] \) or \( v = [\Sigma_2] \) as appropriate.

It follows quickly from this that the fiber sum along a torus of manifolds having simple type produces a manifold of simple type (after sum over rim tori).

It is interesting to compare these results with those in Seiberg-Witten theory. Taubes proved an analogue of Theorem 1.5 in [14], generalizing work of Morgan-Mrowka-Szabó [4]
(in fact, Taubes proves a similar result for an “internal fiber sum” that we do not consider here). The higher-genus case was considered by Morgan, Szabó and Taubes [5], but only under the condition that \(|k| = g - 1\). In this case the sum appearing in Theorem 1.1 is trivial since \(B_{g-1} = \{1\}\), and the result here gives a product formula directly analogous to that of [5]. To our knowledge, no product formulae at the level of generality of Theorems 1.1 and 1.2 have yet appeared in the literature on Seiberg-Witten theory.

The theorems above are proved as particular cases of a very general result on the Ozsváth-Szabó invariants of 4-manifolds obtained by gluing two manifolds along their boundary. To state this result, recall that the construction of the 4-manifold invariant \(\Phi_{M,s}\) is based on the Heegaard Floer homology groups associated to closed spin\(^c\) 3-manifolds \((Y,s)\). These groups have various incarnations; the relevant one for our immediate purpose is denoted \(HF^{-}_{red}(Y,s)\). Below, we recall the construction of Heegaard Floer homology with “twisted” coefficients, whereby homology groups are obtained whose coefficients are modules over the group ring \(R = \mathbb{Z}[t^\pm 1]\) (here and below, ordinary (co)homology is considered with integer coefficients). If \(Y = \partial Z\) is the boundary of an oriented 4-manifold \(Z\), then such a module is provided by

\[ M_Z = \mathbb{Z}[\ker(H^2(Z,\partial Z) \to H^2(Z))], \]

where \(H^1(Y)\) acts by the coboundary homomorphism \(H^1(Y) \to H^2(Z,\partial Z)\). The general product formula alluded to above can be formulated as follows.

**Theorem 1.6.** If \((Z,s)\) is a spin\(^c\) 4-manifold with connected spin\(^c\) boundary \((Y,s_Y)\) and if \(b^+(Z) \geq 1\), then there exists a relative Ozsváth-Szabó invariant \(\Psi_{Z,s}\) which is a linear function

\[ \Psi_{Z,s} : \mathbb{A}(Z) \to HF^{-}_{red}(Y,s_Y; M_Z), \]

well-defined up to multiplication by a unit in \(\mathbb{Z}[H^1(Y)]\).

Furthermore, if \((Z_1,s_1)\) and \((Z_2,s_2)\) are spin\(^c\) 4-manifolds with spin\(^c\) boundary \(\partial Z_1 = (Y,s) = -\partial Z_2\), write \(X = Z_1 \cup_Y Z_2\). Then there exists an \(R_Y\)-sesquilinear pairing

\[ (\cdot, \cdot) : HF^{-}_{red}(Y,s; M_{Z_1}) \otimes_{R_Y} HF^{-}_{red}(-Y,s; M_{Z_2}) \to M_{X,Y}, \]

where \(M_{X,Y} = \mathbb{Z}[K]\) and \(K = \ker(H^2(X) \to H^2(Z_1) \oplus H^2(Z_2))\). Assume that \(b^+(X) \geq 2\). Then the pairing has the property that for any spin\(^c\) structure \(s\) on \(X\) restricting to \(s_i\) on \(Z_i\), we have an equality of group ring elements:

\[ \sum_{h \in K} \Phi_{X,s+h}(\alpha) e^h = (\Psi_{Z_1,s_1}(\alpha_1), \Psi_{Z_2,s_2}(\alpha_2)), \]

up to multiplication by a unit in \(\mathbb{Z}[K]\). Here \(\alpha \in \mathbb{A}(X)\), \(\alpha_1 \in \mathbb{A}(Z_1)\) and \(\alpha_2 \in \mathbb{A}(Z_2)\) are related by inclusion-induced multiplication as before.

To understand the term “\(R_Y\)-sesquilinear,” observe that \(R_Y = \mathbb{Z}[H^1(Y)]\) is equipped with an involution \(r \mapsto \bar{r}\) induced by \(h \mapsto -h\) in \(H^1(Y)\). To say that the pairing in the theorem is sesquilinear means that

\[ (g \xi, \eta) = g(\xi, \eta) = (\xi, g\eta) \]

for \(g \in R_Y\), \(\xi \in HF^{-}_{red}(Y,s; M_{Z_1})\) and \(\eta \in HF^{-}_{red}(-Y,s; M_{Z_2})\).

The utility of Theorem 1.6 is limited somewhat by the difficulty of determining the relative invariants \(\Psi_{Z,s}\) in general. To obtain the fiber sum formulae of Theorems 1.2 and 1.5, we make use of previous work of the authors [2] on the Heegaard Floer homology of 3-manifolds.
of the form $\Sigma \times S^1$ in appropriately twisted coefficient modules, and the relative invariants of $\Sigma \times D^2$, to deduce a relationship between the relative invariants of $M_i \setminus N(E_i)$ and $OS_{M_i}$. Combined with Theorem 1.6, this relationship gives the stated formulae for fiber sums. (The manifold $\Sigma \times D^2$ does not have $b^+ \geq 1$, but we will see that the relative invariant can be defined nevertheless.)

1.1. Examples.

1.1.1. Elliptic surfaces. For $n \geq 2$, let $E(n)$ denote the smooth 4-manifold underlying a simply-connected minimal elliptic complex surface with no multiple fibers and holomorphic Euler characteristic $n$. In [11], Ozsváth and Szabó calculated that $OS_{E(2)} = 1$, meaning that $\Phi_{E(2),s}$ is trivial on all spin$^c$ structures $s$ with $c_1(s) \neq 0$, while if $c_1(s) = 0$ then $\Phi_{E(2),s} = 1$. Implicit in this statement is the fact that $E(2)$ has simple type.

In general, we have that $E(n)$ is diffeomorphic to the fiber sum of $n$ copies of the rational elliptic surface $E(1) = \mathbb{C}P^2 \# 9\mathbb{C}D^2$, summed along copies of the torus fiber $F$ of the elliptic fibration, using the fibration structure to identify neighborhoods of the fibers. In particular, the fiber classes are identified under the fiber sum. Theorem 1.5 then shows that, for example, $OS_{E(6)} = (T - T^{-1})^4$, where $T$ is Poincaré dual to the class of a regular fiber. (In fact, Theorem 1.5 gives this after summing over rim tori using the homomorphism $\rho$ on the left hand side. Arguments based on the adjunction inequality [8, 11], familiar from Seiberg-Witten theory [1], show that only multiples of $T$ can contribute to $OS_{E(n)}$ and therefore application of $\rho$ is unnecessary.) Since $E(6)$ can also be identified with $E(3)\#_F E(3)$, we have that $(T - T^{-1})^4 = (T - T^{-1})^2(\Phi_{E(3)})^2$ so that $OS_{E(3)} = T - T^{-1}$. In general, we see that for $n \geq 2$,

$$OS_{E(n)} = (T - T^{-1})^{n-2}.$$

1.1.2. Higher-genus sums. The elliptic surface $E(n)$ can be realized as the double branched cover of $S^2 \times S^2$, branched along a surface obtained by smoothing the union of 4 parallel copies of $S^2 \times \{pt\}$ and $2n$ copies of $\{pt\} \times S^2$. The projection $\pi_1 : S^2 \times S^2 \to S^2$ to the first factor lifts to an elliptic fibration on $E(n)$, while projection $\pi_2$ on the second factor realizes $E(n)$ as a Lefschetz fibration with typical fiber a surface $\Sigma$ of genus $n - 1$. Note that $\Sigma$ intersects the fiber $F$ of the elliptic fibration in two (positive) points. Let $X_n = E(n)\#_\Sigma E(n)$ denote the fiber sum of two copies of $E(n)$ along $\Sigma$, and suppose $n \geq 3$. We wish to use Theorems 1.1 and 1.2 to calculate the Ozsváth-Szabó invariants of $X_n$.

A useful observation is that $E(n)$ has simple type by the example above. Corollary 1.3 then shows that we can have a nontrivial contribution to $\rho(\Phi_{X_n})$ only when $|k| = g - 1$, i.e., from spin$^c$ structures $s$ with $|c_1(s), [\Sigma]| = 2g - 2 = 2n - 4$. From the preceding example and the fact that $[\Sigma], [F] = 2$, the right-hand side of (3) in the case $|k| = g - 1$ is equal to $\pm 1$, being the product of the invariants arising from $T^{\pm(n-2)}$. Since $T^{\pm(n-2)}$ is equal (up to sign) to the first Chern class $c_1(E(n))$, a convenient way to express these conclusions is that $OS_{X_n} = \pm K \pm K^{-1}$, where $K$ is the canonical class on $X_n$. This formula is true after summing over rim tori, and also possibly the addition of terms pairing trivially with $\Sigma$ since Theorem 1.2 does not address that case.

Note that $X_n$ is diffeomorphic to a complex surface of general type, and therefore this calculation agrees with the corresponding one in Seiberg-Witten theory (ignoring the hypothetical missing terms).
1.2. Organization. The first goal of the paper is to set up enough machinery for the proof of Theorem 1.6. To this end, the next section recalls the definition of Heegaard Floer homology with twisted coefficients from [6] and the corresponding constructions associated to 4-dimensional cobordisms in [8]. In particular, section 2.4 discusses a refinement of the relative grading on Heegaard Floer homology, available with twisted coefficients. Sections 3 and 4 extend other algebraic features of Heegaard Floer homology to the twisted-coefficient relative grading on Heegaard Floer homology, available with twisted coefficients. Sections 3 to 4-dimensional cobordisms in [8]. In particular, section 2.4 discusses a refinement of the homology with twisted coefficients from [6] and the corresponding constructions associated of Theorem 1.6. To this end, the next section recalls the definition of Heegaard Floer oriented 3-manifold Y with “twisted” coefficents. For more details, the reader is referred to [6, 7]. To a closed
1.2. Preliminaries on Twisted Coefficients
2.1. Definitions. We briefly recall the construction of the Heegaard Floer homology groups with “twisted” coefficients. For more details, the reader is referred to [6, 7]. To a closed oriented 3-manifold Y we can associate a pointed Heegaard diagram (Σ, α, β, z) where Σ is a surface of genus g ≥ 1 and α = α₁, ..., αᵣ and β = β₁, ..., βᵣ are sets of attaching circles for the two handlebodies in the Heegaard decomposition. We consider intersection points between the g-dimensional tori Tα = α₁ × · · · × αᵣ and Tβ = β₁ × · · · × βᵣ in the symmetric power \( Sym^g(Σ) \), which we assume intersect transversely. Recall that the basepoint z, chosen away from the αᵢ and βᵢ, gives rise to a map \( s_z : T_α ∩ T_β \to Spin^c(Y) \). Given a spin^c structure \( s \) on Y, the generators for the Heegaard Floer chain complex \( CF^∞(Y, s) \) are pairs \( [x, i] \) where \( i \in \mathbb{Z} \) and \( x \in T_α ∩ T_β \) satisfies \( s_z(x) = s \).

The differential in \( CF^∞ \) counts certain maps \( u : D^2 \to Sym^g(Σ) \) of the unit disk in \( \mathbb{C} \) that connect pairs of intersection points \( x \) and \( y \). That is to say, we consider maps u satisfying the boundary conditions:

\[
\begin{align*}
  u(e^{iθ}) \in T_α & \text{ for } \cos θ ≥ 0 & u(i) = y \\
  u(e^{iθ}) \in T_β & \text{ for } \cos θ ≤ 0 & u(-i) = x.
\end{align*}
\]

For \( g > 2 \) we let \( π_2(x, y) \) denote the set of homotopy classes of such maps; for \( g = 2 \) we let \( π_2(x, y) \) be the quotient of the set of such homotopy classes by a further equivalence, the details of which need not concern us (see [6]).

There is a topological obstruction to the existence of any such disk connecting \( x \) and \( y \), denoted \( ε(x, y) ∈ H_1(Y; Z) \). To any homotopy class \( φ ∈ π_2(x, y) \) we can associate the quantity \( n_z(φ) \), being the algebraic intersection number between \( φ \) and the subvariety \( \{z\} × Sym^{g−1}(Σ) \). The following is a basic fact in Heegaard Floer theory:

Proposition 2.1 ([6]). Suppose \( g > 1 \) and let \( x, y \in T_α ∩ T_β \). If \( ε(x, y) \neq 0 \) then \( π_2(x, y) \) is empty, while if \( ε(x, y) = 0 \) then there is an affine isomorphism

\[
π_2(x, y) = \mathbb{Z} ⊕ H^1(Y; Z),
\]

such that the projection \( π_2(x, y) → \mathbb{Z} \) is given by the map \( n_z \).

There is a natural “splicing” of homotopy classes

\[
π_2(x, y) ∗ π_2(y, z) → π_2(x, z),
\]
as well as an action

\[ \pi_2' \left( Sym^g(\Sigma_g) \right) \times \pi_2(x, y) \to \pi_2(x, y), \]

where \( \pi_2' \) denotes the second homotopy group divided by the action of the fundamental group. \((\text{For } g > 1, \pi_2'(Sym^g(\Sigma_g)) \cong \mathbb{Z}, \text{ generated by a class } S \text{ with } n_z(S) = 1. \) When \( g > 2, \pi_2'(Sym^g(\Sigma_g)) = \pi_2(Sym^g(\Sigma_g)). \)} The isomorphism in the above proposition is affine in the sense that it respects the splicing action by \( \pi_2(x, y). \)

The ordinary “untwisted” version of Heegaard Floer homology takes \( CF^\infty \) to be generated (over \( \mathbb{Z} \)) by pairs \([x, i]\) as above, equipped with a boundary map such that the coefficient of \([y, j]\) in the boundary of \([x, i]\) is the number of pseudo-holomorphic maps in all homotopy classes \( \phi \in \pi_2(x, y) \) having \( n_z(\phi) = i - j. \) The twisted version is similar, but where one keeps track of all possible homotopy data associated to \( \phi. \) In light of the above proposition, this means that we should form a chain complex freely generated by intersection points \( x \) as a module over the group ring of \( \mathbb{Z} \oplus H^1(Y; \mathbb{Z}), \) or equivalently by pairs \([x, i]\) over the group ring of \( H^1(Y; \mathbb{Z}). \) Following \([7,\) we make the following definition:

**Definition 2.2.** An additive assignment for the diagram \((\Sigma, \alpha, \beta, z)\) is a collection of functions 

\[ A_{x,y} : \pi_2(x, y) \to H^1(Y; \mathbb{Z}) \]

that satisfies

1. \( A_{x,z}(\phi * \psi) = A_{x,y}(\phi) + A_{y,z}(\psi) \) whenever \( \phi \in \pi_2(x, y) \) and \( \psi \in \pi_2(y, z). \)
2. \( A_{x,y}(S * \phi) = A_{x,y}(\phi) \) for \( S \in \pi_2'(Sym^g(\Sigma_g)). \)

We will drop the subscripts from \( A_{x,y} \) whenever possible. It is shown in \([7,\) how a certain finite set of choices (a “complete set of paths”) gives rise to an additive assignment in the above sense.

**Definition 2.3.** Let \((\Sigma, \alpha, \beta, z)\) be a pointed Heegaard diagram for \( Y \) and \( s \in Spin^c(Y). \) Fix an additive assignment \( A \) for the diagram. The twisted Heegaard Floer chain complex \( CF^\infty(Y, s; \mathbb{Z}[H^1(Y; \mathbb{Z})]) \) is the module freely generated over \( \mathbb{Z}[H^1(Y; \mathbb{Z})] \) by pairs \([x, i]\), with differential \( \partial^\infty \) given by

\[ \partial^\infty[x, i] = \sum_{y \in T_n \cap T_\beta} \sum_{\phi \in \pi_2(x, y), \mu(\phi) = 1} #M(\phi) \cdot e^{A(\phi)}[y, i - n_z(\phi)]. \]

Here \( M(\phi) \) denotes the space of holomorphic disks in the homotopy class \( \phi, \) where “holomorphic” is defined relative to an appropriately generic path of almost-complex structure on \( Sym^g(\Sigma_g). \) For such a path, \( M(\phi) \) is a smooth manifold of dimension given by a Maslov index \( \mu(\phi). \) There is an action of \( \mathbb{R} \) on \( M(\phi) \) by reparametrization of the disk, and \( \hat{M}(\phi) \) denotes the quotient of \( M(\phi) \) by this action. When \( \mu(\phi) = 1, \hat{M}(\phi) \) is a compact, zero-dimensional manifold. An appropriate choice of “coherent orientation system” serves to orient the points of \( \hat{M}(\phi) \) in this case, and \( #\hat{M}(\phi) \) denotes the signed count of these points. It is shown in \([6,7,\) that under appropriate admissibility hypotheses on the diagram \((\Sigma, \alpha, \beta, z)\) the chain homotopy type of \( CF^\infty(Y, s; \mathbb{Z}[H^1(Y)]) \) is an invariant of \((Y, s).\)
As in the introduction, in much of what follows we will write $R_Y$ for the ring $\mathbb{Z}[H^1(Y;\mathbb{Z})]$, or simply $R$ when the underlying 3-manifold is apparent from context. Note that by choosing a basis for $H^1(Y;\mathbb{Z})$ we can identify $R$ with the ring of Laurent polynomials in $b_1(Y)$ variables.

By following the usual constructions of Heegaard Floer homology, we obtain other variants of the above with coefficients in $R_Y$: namely by considering only generators $[x, i]$ with $i < 0$ we obtain a subcomplex $CF^-(Y, s; R)$ whose quotient complex is $CF^+(Y, s; R)$, with associated homology groups $HF^-$ and $HF^+$ respectively. There is an action $U : [x, i] \mapsto [x, i - 1]$ on $CF^\infty$ as usual; the kernel of the induced action on $CF^+$ is written $\hat{CF}$ with homology $\hat{HF}(Y, s; R)$. There is a relative grading on $CF$ with respect to which $U$ decreases degree by 2; we will discuss gradings further in section 2.4.

Finally, given any module $M$ for $R_Y$ we can form Heegaard Floer homology with coefficients in $M$ by taking the homology of the complex $\hat{CF}$ as usual; the kernel of the induced action on $CF^+$ is written $\hat{CF}$ with homology $\hat{HF}(Y, s; R)$. There is a relative grading on $CF$ with respect to which $U$ decreases degree by 2; we will discuss gradings further in section 2.4.

2.2. Twisted cobordism invariants. We now sketch the construction and main properties of twisted-coefficient Heegaard Floer invariants associated to cobordisms, which can be found in greater detail in [8]. Recall that if $W : Y_1 \to Y_2$ is an oriented 4-dimensional cobordism and $M$ is a module for $R_1 := R_{Y_1} = \mathbb{Z}[H^1(Y_1;\mathbb{Z})]$, then there is an induced module $M(W)$ for $R_2 = R_{Y_2}$ defined as follows. Let

$$K(W) = \ker(H^2(W, \partial W; \mathbb{Z}) \to H^2(W; \mathbb{Z}))$$

be the kernel of the map in the long exact sequence for the pair $(W, \partial W)$; then $\mathbb{Z}[K(W)]$ is a module for $R_1$ and $R_2$ via the coboundary maps $H^1(Y_1; \mathbb{Z}) \to K(W) \subset H^2(W, \partial W)$. Define

$$M(W) = M \otimes_{R_1} \mathbb{Z}[K(W)].$$

Then $M(W)$ is a module for $R_2$ in the obvious way.

Ozsváth and Szabó show in [8] how to associate to a cobordism $W$ as above with spin$^c$ structure $s$ a homomorphism

$$F^\circ_{W, s} : HF^\circ(Y_1, s_1; M) \to HF^\circ(Y_2, s_2; M(W))$$

(where $s_i$ denotes the restriction of $s$ to $Y_i$, and $\circ$ indicates a map between each of the varieties of Heegaard Floer homology, respecting the long exact sequences relating them). This is defined as a composition

$$F^\circ_W = E^\circ \circ H^\circ \circ G^\circ,$$

where $G^\circ$ is associated to the 1-handles in $W$, $H^\circ$ to the 2-handles, and $E^\circ$ to the 3-handles. Note that the coefficient module remains unchanged by cobordisms consisting of 1- or 3-handle additions. Indeed, such cobordisms induce homomorphisms in an essentially formal way, so we simply refer the reader to [8] for the definition of $E^\circ$ and $G^\circ$.

Suppose that $W$ is a cobordism consisting of 2-handle additions, so that we can think of $W$ as associated to surgery on a framed link $L \subset Y_1$. In this situation, Ozsváth and Szabó construct a “Heegaard triple” associated to $W$, $(\Sigma_g, \alpha, \beta, \gamma, z)$. This diagram describes three
3-manifolds $Y_{\alpha\beta}$, $Y_{\beta\gamma}$ and $Y_{\alpha\gamma}$ obtained by using the indicated circles on $\Sigma$ as attaching circles, such that

$$Y_{\alpha\beta} = Y_1, \quad Y_{\beta\gamma} = \#^k S^1 \times S^2, \quad Y_{\alpha\gamma} = Y_2,$$

where $k$ is the genus of $\Sigma$ minus the number of components of $L$. In fact the diagram $(\Sigma, \alpha, \beta, \gamma, z)$ describes a 4-manifold $X_{\alpha\beta\gamma}$ in a natural way, whose boundaries are the three manifolds above. Furthermore, in the current situation, $X_{\alpha\beta\gamma}$ is obtained from $W$ by removing the regular neighborhood of a 1-complex (see [8]).

We can arrange that the top-dimensional generator of $HF^{\leq 0}(Y_{\beta\gamma}, s_0; \mathbb{Z}) \cong \Lambda^* H^1(Y_{\beta\gamma}; \mathbb{Z}) \otimes \mathbb{Z}[U]$ is represented by an intersection point $\Theta \in T_\beta \cap T_\gamma$ (here $s_0$ denotes the spin$^c$ structure on $\#^k S^1 \times S^2$ having $c_1(s_0) = 0$). The map $F^c$ is defined by counting holomorphic triangles, with the aid of another additive assignment. To describe this, suppose $x \in T_\alpha \cap T_\beta$, $y \in T_\beta \cap T_\gamma$, and $w \in T_\alpha \cap T_\gamma$ are intersection points arising from a Heegaard triple $(\Sigma, \alpha, \beta, \gamma, z)$. Let $\Delta$ denote a standard 2-simplex, and write $\pi_2(x, y, w)$ for the set of homotopy classes of maps $u : \Delta \to Sym^\theta(\Sigma)$ that send the boundary arcs of $\Delta$ into $T_\alpha$, $T_\beta$, and $T_\gamma$ respectively, under a clockwise ordering of the boundary arcs $e_\alpha$, $e_\beta$, and $e_\gamma$ of $\Delta$, and such that

$$u(e_\alpha \cap e_\beta) = x, \quad u(e_\beta \cap e_\gamma) = y, \quad u(e_\alpha \cap e_\gamma) = w.$$

Again there is a topological obstruction $\epsilon(x, y, w) \in H_1(X_{\alpha\beta\gamma}; \mathbb{Z})$ that vanishes if and only if $\pi_2(x, y, w)$ is nonempty. The analogue of Proposition 2.1 in this context is the following.

**Proposition 2.4** ([8]). Let $(\Sigma, \alpha, \beta, \gamma, z)$ be a pointed Heegaard triple as above, and $X_{\alpha\beta\gamma}$ the associated 4-manifold. Then whenever $\epsilon(x, y, w) = 0$ we have an (affine) isomorphism

$$\pi_2(x, y, w) \cong \mathbb{Z} \oplus H_2(X_{\alpha\beta\gamma}; \mathbb{Z})$$

where the projection to $\mathbb{Z}$ is given by $\psi \mapsto n_z(\psi)$.

There is an obvious “splicing” action on homotopy classes of triangles by disks on each corner; the above identification respects this action.

Recall from [7] that the basepoint $z$ gives rise to a map

$$s_z : \coprod_{x, y, w} \pi_2(x, y, w) \to \text{Spin}^c(X_{\alpha\beta\gamma}),$$

such that triangles $\psi \in \pi_2(x, y, w)$ and $\psi' \in \pi_2(x', y', w')$ have $s_z(\psi) = s_z(\psi')$ if and only if there exist disks $\phi_x \in \pi_2(x, x')$, $\phi_y \in \pi_2(y, y')$ and $\phi_w \in \pi_2(w, w')$ such that $\psi' = \psi + \phi_x + \phi_y + \phi_w$. In this case $\psi$ and $\psi'$ are said to be spin$^c$ equivalent. Note that in case $(\Sigma, \alpha, \beta, \gamma, z)$ describes a 2-handle cobordism $W$ as previously, we can think of $s_z$ as a function

$$s_z : \coprod_{x, w} \pi_2(x, \Theta, w) \to \text{Spin}^c(W).$$

**Definition 2.5.** An additive assignment for a Heegaard triple $(\Sigma, \alpha, \beta, \gamma, z)$ describing a 2-handle cobordism $W : Y_1 \to Y_2$ as above is a function

$$A_W : \coprod_{s \in \text{Spin}^c(W)} s_z^{-1}(s) \to K(W)$$
obtained in the following manner. For a fixed \( \psi_0 \in s_2^{-1}(s) \), let \( \psi = \psi_0 + \phi_{\alpha} + \phi_{\beta} + \phi_{\gamma} \) be an arbitrary element of \( s_2^{-1}(s) \). Then set
\[
A_W(\psi) = \delta(A_1(\phi_{\alpha}) + A_2(\phi_{\gamma})),
\]
where \( A_i \) are additive assignments for \( Y_i \) and \( \delta : H^1(\partial W; Z) \to H^2(W, \partial W; Z) \) is the Mayer-Vietoris coboundary.

We are now in a position to define the map on Floer homology induced by \( W \) (given additive assignments on \( Y_1, Y_2, \) and \( W \)). We again refer to [7, 8] for the details required to make full sense of the following.

**Definition 2.6.** For a triple \((\Sigma, \alpha, \beta, \gamma, z)\) describing a 2-handle cobordism \( W \) with spin\(^c\) structure \( s \), we define
\[
F_{W,s}^\circ : HF^\circ(Y_1, s_1; M) \to HF^\circ(Y_2, s_2; M(W)),
\]
where \( s_i = s|_{Y_i} \), to be the map induced on homology by the chain map
\[
[x, i] \mapsto \sum_{w \in T_0 \cap T_1} \sum_{\psi \in \pi_2(x, \Theta, w)} \#M(\psi) \cdot [w, i - n_z(\psi)] \otimes e^{A_W(\psi)}.
\]
Here \( \mu(\psi) \) denotes the expected dimension of the moduli space \( M(\psi) \) of pseudo-holomorphic triangles in the homotopy class \( \psi \), and \( \#M(\psi) \) indicates the signed count of points in a compact oriented 0-dimensional manifold.

We should note that while the Floer homology \( HF^\circ(Y, s; M) \) does not depend on the additive assignment \( A_Y \), the map \( F_{W,s}^\circ \) does depend on the choice of \( A_W \) as in definition 2.5 through the reference triangle \( \psi_0 \). Changing this choice has the effect of pre- (post-) composing \( F_W \) with the action of an element of \( H^1(Y_1; Z) \) (resp \( H^1(Y_2; Z) \)), which in turn act in \( M(W) \) via the Mayer-Vietoris coboundary. Likewise the generator \( \Theta \) is determined only up to sign, so that \( F_W \) has a sign indeterminacy as well. Following [8], we let \( [F_{W,s}^\circ] \) denote the orbit of \( F_{W,s}^\circ \) under the action of \( H^1(Y_1; Z) \oplus H^1(Y_2; Z) \).

2.3. Composition law. A key advantage to using twisted coefficient modules for Heegaard Floer homology is the availability of a refined composition law in this situation. To describe this, we must first understand the behavior of the coefficient modules themselves under composition of cobordisms. The following lemma will be useful in formulating results to come; here and to follow, we take (co)homology with integer coefficients.

**Lemma 2.7.** Let \( W = W_1 \cup Y_1 W_2 \) be the composition of two cobordisms \( W_1 : Y_0 \to Y_1 \) and \( W_2 : Y_1 \to Y_2 \). Define
\[
K(W, Y_1) = \ker[\rho_1 \oplus \rho_2 : H^2(W, \partial W) \to H^2(W_1) \oplus H^2(W_2)],
\]
where \( \rho_i \) denotes the restriction map \( H^2(W, \partial W) \to H^2(W_i) \). Then
\[
\mathbb{Z}[K(W_1)] \otimes_{\mathbb{Z}[H^1(Y_1)]} \mathbb{Z}[K(W_2)] \cong \mathbb{Z}[K(W, Y_1)]
\]
as modules over \( \mathbb{Z}[H^1(Y_0)] \) and \( \mathbb{Z}[H^1(Y_2)] \).
Proof. We have

$$\mathbb{Z}[K(W_1)] \otimes_{\mathbb{Z}[H^1(Y_1)]} \mathbb{Z}[K(W_2)] \cong \mathbb{Z} \left[ \frac{K(W_1) \oplus K(W_2)}{H^1(Y_1)} \right],$$

so the claim amounts to exhibiting an isomorphism

$$\frac{K(W_1) \oplus K(W_2)}{H^1(Y_1)} \cong K(W, Y_1).$$

To see this, consider the diagram

$$\begin{array}{ccc}
H^1(Y_1) & \longrightarrow & H^2(W_1, \partial W_1) \oplus H^2(W_2, \partial W_2) \\
& \Downarrow f & \longrightarrow \\
& \rho_1 \oplus \rho_2 & \longrightarrow \\
& H^2(W_1) \oplus H^2(W_2), \\
\end{array}$$

where the horizontal row is (the Poincaré dual of) the Mayer-Vietoris sequence. Write

$$i_* : H^2(W_1, \partial W_1) \rightarrow H^2(W, \partial W) \quad \text{and} \quad j_* : H^2(W_2, \partial W_2) \rightarrow H^2(W, \partial W)$$

for the components of $f$; then it is not hard to see that

$$\rho_1 \circ i_* : H^2(W_1, \partial W_1) \rightarrow H^2(W_1) \quad \text{and} \quad \rho_2 \circ j_* : H^2(W_2, \partial W_2) \rightarrow H^2(W_2)$$

agree with the maps induced by inclusion, while

$$\rho_2 \circ i_* = 0 \quad \text{and} \quad \rho_1 \circ j_* = 0.$$

From this it is easy to deduce that $f^{-1}(K(W, Y_1)) = K(W_1) \oplus K(W_2)$, from which the lemma follows. \qed

Remark 2.8. If $W$ is a cobordism between homology spheres, or more generally if $H^2(W, \partial W) \rightarrow H^2(W)$ is an isomorphism, then there is an identification

$$K(W, Y_1) = \ker[H^2(W) \rightarrow H^2(W_1) \oplus H^2(W_2)],$$

the kernel of the restriction map in the ordinary Mayer-Vietoris sequence in cohomology. In this case if $s_1$ and $s_2$ are spin$^c$ structures on $W_1$ and $W_2$, then $K(W, Y_1)$ parametrizes spin$^c$ structures $s$ on $W$ such that $s|_{W_i} = s_i$ (when that set is nonempty). In the case of a closed 4-manifold $X$, the module $M_{X,Y}$ of the introduction is simply $\mathbb{Z}[K(W, Y)]$ where $W$ is obtained from $X$ by removing a 4-ball on each side of $Y$.

When regarding $W$ as a single cobordism the group relevant to twisted coefficient modules is $K(W)$, while if $W = W_1 \cup W_2$ is viewed as a composite the coefficient modules change by tensor product with the group ring of $K(W, Y_1)$ (in light of the lemma above). By
commutativity of the diagram

\[
\begin{array}{ccc}
H^2(W, \partial W) & \longrightarrow & H^2(W) \\
\rho_1 \oplus \rho_2 & \downarrow & \\
H^2(W_1) \oplus H^2(W_2),
\end{array}
\]

there is a natural inclusion \( \iota : K(W) \rightarrow K(W, Y_1) \). This gives rise to a projection map

\[
\Pi : \mathbb{Z}[K(W, Y_1)] \rightarrow \mathbb{Z}[K(W)],
\]

namely

\[
\Pi(e^w) = \begin{cases} 
e^w & \text{if } w = \iota(v) \text{ for some } v \\ 0 & \text{otherwise} \end{cases}
\]

(c.f. [8]).

Thus, if \( M \) is a module for \( \mathbb{Z}[H^1(Y_0)] \) we obtain a map

\[
\Pi_M : M(W_1)(W_2) \rightarrow M(W)
\]

by tensor product of \( \Pi \) with the identity under the identifications

\[
M(W_1)(W_2) = M \otimes_{\mathbb{Z}[H^1(Y_0)]} \mathbb{Z}[K(W, Y_1)] \quad \text{and} \quad M(W) = M \otimes_{\mathbb{Z}[H^1(Y_0)]} \mathbb{Z}[K(W)].
\]

The refined composition law for twisted coefficients can be stated as follows.

**Theorem 2.9** (Theorem 3.9 of [8]). Let \( W = W_1 \cup_{Y_1} W_2 \) be a composite cobordism as above with spin\(^c\) structure \( s \). Write \( s_i = s|_{W_i} \). Then there are choices of representatives for the various maps involved such that

\[
[F^\circ_{W,s}] = [\Pi_M \circ F^\circ_{W_2,s_2} \circ F^\circ_{W_1,s_1}].
\]

More generally, if \( h \in H^1(Y_1; \mathbb{Z}) \) then for these choices we have

\[
[F^\circ_{W,s+\delta h}] = [\Pi_M \circ F^\circ_{W_2,s_2} \circ e^h \cdot F^\circ_{W_1,s_1}],
\]

where \( \delta h \) is the image of \( h \) under the Mayer-Vietoris coboundary \( H^1(Y_1) \rightarrow H^2(W) \).

We should also remark that for a cobordism \( W : Y_1 \rightarrow Y_2 \) with spin\(^c\) structure \( s \) the map

\[
F^\circ_{W,s} : HF^\circ(Y_1, s_1; \mathbb{Z}) \rightarrow HF^\circ(Y_2, s_2; \mathbb{Z})
\]

in untwisted Floer homology can be obtained from the twisted-coefficient map

\[
HF^\circ(Y_1, s_1; \mathbb{Z}) \rightarrow HF^\circ(Y_2, s_2; \mathbb{Z}(W))
\]

(here \( \mathbb{Z}(W) \) is the module \( M(W) \) with \( M = \mathbb{Z} \), namely \( \mathbb{Z}(W) = \mathbb{Z} \otimes_{\mathbb{Z}[H^1(Y_1)]} \mathbb{Z}[K(W)] = \mathbb{Z}[\ker(H^2(W, Y_2) \rightarrow H^2(W))]) by composition with the map \( \epsilon_* \) induced in homology by the homomorphism

\[
\epsilon : \mathbb{Z}(W) \rightarrow \mathbb{Z}
\]

of coefficient modules that sends each element of \( \ker(H^2(W, Y_2) \rightarrow H^2(W)) \) to 1.
2.4. **Refined relative gradings.** The \(\mathbb{Z}\)-coefficient version of Heegaard Floer homology is naturally a relatively cyclically graded theory, in general. This means that if \(S = \{[x, i] | s_z(x) = s\}\) denotes the natural generating set for \(CF^\infty(Y, s; \mathbb{Z})\) then there is a map

\[
\text{gr} : S \times S \to \mathbb{Z}/\mathfrak{d}(s)\mathbb{Z},
\]

where

\[
\mathfrak{d}(s) = \gcd\{\langle c_1(s), h \rangle | h \in H_2(Y; \mathbb{Z})\}
\]

is the \textit{divisibility} of \(c_1(s)\) (or by abuse of language, of \(s\) itself). The differential in \(CF^\infty\) has degree \(-1\) with respect to this grading, while the endomorphism \(U\) has degree \(-2\).

In the case of fully twisted coefficients (coefficients in \(\mathbb{Z}[H^1(Y)]\), Ozsváth and Szabó [6] observe that there is a lift of this cyclic grading to a relative \(\mathbb{Z}\)-grading. Here we provide an interpolation between these situations for arbitrary coefficient modules \(M\).

**Definition 2.10.** Let \(Y\) be a 3-manifold, and suppose \(M\) is a module over \(\mathbb{Z}[H^1(Y; \mathbb{Z})]\). The stabilizer \(Z_M\) of \(M\) is the kernel of the group homomorphism \(H^1(Y; \mathbb{Z}) \to \text{Aut}(M)\). More explicitly,

\[
Z_M = \{x \in H^1(Y; \mathbb{Z}) | xm = m \text{ for all } m \in M\}.
\]

**Definition 2.11.** Let \(Y\) be a closed oriented 3-manifold and \(M\) a module for \(\mathbb{Z}[H^1(Y; \mathbb{Z})]\). The \(M\)-divisibility \(\mathfrak{d}_M(\alpha)\) of a class \(\alpha \in H^2(Y; \mathbb{Z})\) is the quantity

\[
\mathfrak{d}_M(\alpha) = \gcd\{\langle \alpha \cup z, [Y] \rangle | z \in Z_M\}.
\]

If \(s\) is a spin\(^c\) structure on \(Y\), the \(M\)-divisibility of \(s\), \(\mathfrak{d}_M(s)\), is the \(M\)-divisibility of \(c_1(s)\).

In this notation, the (ordinary) divisibility of \(s\) is \(\mathfrak{d}_Z(s)\), where \(Z\) is the trivial \(\mathbb{Z}[H^1(Y; \mathbb{Z})]\)-module. Indeed, in this case \(Z_M = Z = H^1(Y; \mathbb{Z})\), and (5) agrees with (6).

Now suppose \([x, i]\) and \([y, j]\) are generators for \(CF^\infty(Y, s; M)\), for an arbitrary \(\mathbb{Z}[H^1(Y)]\)-module \(M\). We define

\[
\text{gr}([x, i], [y, j]) \in \mathbb{Z}/\mathfrak{d}_M(s)\mathbb{Z}
\]

by the formula

\[
\text{gr}([x, i], [y, j]) = \mu(\phi) - 2n_z(\phi) + 2(i - j),
\]

where \(\phi \in \pi_2(x, y)\) is any disk such that \(A_{x,y}(\phi) \in Z_M\).

To see that (7) gives a well-defined relative grading modulo \(\mathfrak{d}_M(s)\), note that any two disks joining the points \(x\) and \(y\) differ by a periodic domain \(P\), corresponding to a homology class \(H(P) \in H_2(Y; \mathbb{Z})\) (or, via Poincaré duality, a cohomology class \(H(P) \in H^1(Y; \mathbb{Z})\)). If the pair of disks have image in \(Z_M\) under the additive assignment \(A\) then so does \(P\). Since an additive assignment respects the natural identification of periodic domains with (co)homology classes, it follows that \(\mathfrak{d}_M(s)\) divides \(\langle c_1(s), H(P) \rangle = \langle c_1(s) \cup H(P), [Y] \rangle\). But according to Theorem 4.9 of [6], this quantity is precisely the difference in Maslov indices between the two disks.

**Example 2.12.** 1. If \(M = R_Y = \mathbb{Z}[H^1(Y)]\), then \(H^1(Y)\) is a subgroup of \(\text{Aut}(M)\), so \(Z_M = 0\). Therefore \(\mathfrak{d}_M(s) = 0\) for all \(s\), meaning that \(CF^\infty(Y, s; \mathbb{Z}[H^1(Y)])\) acquires a relative \(\mathbb{Z}\) grading for any \(s\).
Let $Y = \Sigma \times S^1$ be the product of a surface $\Sigma$ of genus $g$ with a circle, and let $M = L(t)$ be the ring of Laurent polynomials over $\mathbb{Z}$ in a single variable $t$. We make $L(t)$ into a module over $\mathbb{Z}[H^1(Y)]$ using the map $\mathbb{Z}[H^1(Y)] \to L(t)$ given by

$$\alpha \mapsto t^{\langle \alpha, [S^1] \rangle}$$

for $\alpha \in H^1(Y)$: then $Z_{L(t)}$ consists of those classes $\alpha$ with $\langle \alpha, [S^1] \rangle = 0$. (One can think of $L(t)$ as the group ring of the subgroup of $H^1(Y)$ generated by the Poincaré dual of $[\Sigma \times pt]$.)

Now let $k \in \mathbb{Z}$. Then if $s \in \text{Spin}^c(Y)$ is the spin$^c$ structure with first Chern class $c_1(s) = 2k \cdot P.D.[S^1]$ we have that $\langle c_1(s) \cup \alpha, [Y] \rangle = 0$ for all $\alpha \in Z_{L(t)}$, so that the Heegaard Floer homology groups $HF^0(Y, s; L(t))$ carry a relative $\mathbb{Z}$-grading.

We close this section with an observation regarding the naturality of the refined gradings under cobordisms.

**Proposition 2.13.** Suppose $W : Y_0 \to Y_1$ is a cobordism with spin$^c$ structure $s$, and write $s_i = s|_{Y_i}$. For any module $M$ over $\mathbb{Z}[H^1(Y_0)]$, we have that

$$\partial_M(s_0) \text{ divides } \partial_{M(W)}(s_1).$$

In particular, if $\partial_M(s_0) = 0$, then $\partial_{M(W)}(s_1) = 0$ as well.

**Proof.** We need to calculate the stabilizer of $M(W)$ in $H^1(Y_1; \mathbb{Z})$. One easily finds

$$(8) \quad Z_{M(W)} = \{ z_1 \in H^1(Y_1) \mid \exists z_0 \in Z_M \text{ with } (z_0, z_1) \in \ker(H^1(\partial W) \to H^2(W, \partial W)) \}$$

using the identification $H^1(\partial W) = H^1(Y_0) \oplus H^1(Y_1)$.

Fix $z_1 \in Z_{M(W)}$ and let $i_0$ and $i_1$ denote the inclusion maps of $Y_0$ and $Y_1$ into $W$ respectively. Then from (8) we can find a class $z_0 \in Z_M$ with $\delta z_1 - \delta z_0 = 0$, where $\delta$ denotes the coboundary $H^1(Y_1) \to H^1(W, \partial W)$. Thus

$$c_1(s_1) \cup z_1 = i_1^* c_1(s) \cup z_1$$

$$= c_1(s) \cup \delta z_1$$

$$= c_1(s) \cup (\delta z_1 - \delta z_0) + c_1(s) \cup \delta z_0$$

$$= c_1(s_0) \cup z_0.$$ 

If $\partial_M(s_0) = 0$ then since $z_0 \in Z_M$ this last expression vanishes, proving that $\partial_{M(W)}(s_1) = 0$. In general, we have that $\partial_{M(W)}(s_0) \text{ divides } c_1(s_0) \cup z_0$, which by the calculation above implies $\partial_{M}(s_0) \text{ divides } \partial_{M(W)}(s_1)$. \qed

**Corollary 2.14.** If $(W, s)$ is any spin$^c$ cobordism between $S^3$ (or any rational homology sphere) and a 3-manifold $Y$, then in the induced spin$^c$ structure on $Y$, the Heegaard Floer homology with coefficients in the module induced by $W$ carries a relative $\mathbb{Z}$ grading. \qed
3. Pairings and Duality

In [8], Ozsváth and Szabó defined a pairing
\[ \langle \cdot, \cdot \rangle : HF^+(Y, s; \mathbb{Z}) \otimes HF^-(\neg Y, s; \mathbb{Z}) \to \mathbb{Z} \]
on Floer homology (where \( \neg Y \) denotes the manifold \( Y \) with the reversed orientation), with respect to which cobordism-induced maps satisfy a certain duality. Here we extend this pairing to Floer homology with twisted coefficients and prove a corresponding duality. As usual, we use the notation \( R_Y \), or simply \( R \), for the ring \( \mathbb{Z}[\pi_1(Y; \mathbb{Z})] \).

First, recall that \( R \) is equipped with an involution \( r \mapsto \bar{r} \) induced by \( h \mapsto -h \) on \( \pi_1(Y; \mathbb{Z}) \).

If \( M \) is an \( R \)-module, we let \( \overline{M} \) denote the group \( M \) equipped with the "conjugate" module structure in which module multiplication is given by \( r \otimes m \mapsto \bar{r} \cdot m \).

Recall also that if \((\Sigma, \alpha, \beta, z)\) is a pointed Heegaard diagram for \( Y \), then \((\neg \Sigma, \alpha, \beta, z)\) describes \( \neg Y \), and the map \( s_z \) is invariant under this change of orientation.

Definition 3.1. Define a pairing\[ \langle \cdot, \cdot \rangle : CF^\infty(Y, s; R) \otimes_R CF^\infty(\neg Y, s; \overline{R}) \to R \]
as follows: for generators \([x, i] \in CF^\infty(Y, s; R)\) and \([y, j] \in CF^\infty(\neg Y, s; R)\) set
\[ \langle [x, i], [y, j] \rangle = \begin{cases} 1 & \text{if } x = y \text{ and } j = -i - 1 \\ 0 & \text{otherwise}. \end{cases} \]
The desired pairing is obtained by extending by \( R \)-linearity.

We must check that this definition has the desired properties:

Lemma 3.2. For any \( \xi \in CF^\infty(Y, s; R) \), \( \eta \in CF^\infty(\neg Y, s; R) \), we have
\[ \langle \partial^\infty \xi, \eta \rangle = \langle \xi, \partial^\infty \eta \rangle \]
\[ \langle U \xi, \eta \rangle = \langle \xi, U \eta \rangle. \]

Proof. This is much like the proof of the corresponding fact in untwisted Floer homology [8], but we must be more careful with the coefficients. Observe that composition with the reflection \( r : D^2 \to D^2 \) across the real axis gives a map \( \pi_2(x, y) \to \pi_2(y, x) \) that exchanges \( J \)-holomorphic disks in \( Sym^\theta(\Sigma) \) with \( -J \)-holomorphic disks in \( Sym^\theta(\neg \Sigma) \); in other words
\[ M_{-\Sigma}(\phi \circ r) = M_{\Sigma}(\phi) \]
for \( \phi \in \pi_2(x, y) \).

Furthermore, if \( A \) is an additive assignment for \((\Sigma, \alpha, \beta, z)\) then we can think of \( A \) as also giving an additive assignment for \((\neg \Sigma, \alpha, \beta, z)\). For \( \phi \in \pi_2(x, y) \) we have that \( \phi \ast (\phi \circ r) \) is homotopic to a constant map, from which it follows that
\[ A_{y,x}(\phi \circ r) = -A_{x,y}(\phi). \]
Since \( n^\Sigma_z(\phi) = n^-\Sigma_z(\phi \circ r) \), we have
\[
\partial^\infty[y,j] = \sum_{\phi \in \pi_2(y,w)} \# \widehat{M}^-\Sigma(\phi)[w, j - n^-\Sigma_z(\phi)] \otimes e^{A(\phi)}
\]
\[
= \sum_{\phi \in \pi_2(w,y)} \# \widehat{M}^-\Sigma(r(\phi))[w, j - n^-\Sigma_z(r(\phi))] \otimes e^{A(r(\phi))}
\]
\[
= \sum_{\phi \in \pi_2(w,y)} \# \widehat{M}^-\Sigma(\phi)[w, j - n^-\Sigma_z(\phi)] \otimes e^{-A(\phi)}.
\]
From this it follows (using the conjugate module structure on the second factor) that
\[
\langle [x, i], \partial^\infty[y, j] \rangle = \sum_{\phi \in \pi_2(x,y)} \sum_{\mu(\phi) = 1} \sum_{n_z(\phi) = i + j + 1} \# \widehat{M}(\phi)e^{A(\phi)}
\]
\[
= \langle \partial^\infty[x, i], [y, j] \rangle.
\]
The first claim of the lemma follows from this, while the second is obvious. \( \square \)

Thus we obtain a pairing on homology
\[
HF^+(Y, s; R) \otimes_R HF^-(Y, s; R) \rightarrow R.
\]

More generally, suppose \( M \) and \( N \) are \( R \)-modules: we can extend the construction above to a pairing between \( HF^+(Y, s; M) \) and \( HF^-(Y, s; N) \). To this end, define
\[
\langle \cdot, \cdot \rangle : CF^\infty(Y, s; M) \otimes_R CF^\infty(Y, s; N) \rightarrow M \otimes_R N
\]
on generators by
\[
\langle [x, i] \otimes m, [y, j] \otimes n \rangle = \langle [x, i], [y, j] \rangle \cdot m \otimes n,
\]
where the pairing on the right is the universal one just defined. It follows from the calculation above that the pairing descends to homology:
\[
HF^+(Y, s; M) \otimes_R HF^-(Y, s; N) \rightarrow M \otimes_R N.
\]

Before proceeding to the statement of duality, we make the following observation.

**Lemma 3.3.** Let \( W : Y_1 \rightarrow Y_2 \) be a cobordism between connected oriented 3-manifolds. Write \( R_i = \mathbb{Z}[H^1(Y_i; \mathbb{Z})] \), and suppose that \( M_i \) is a module over \( R_i \) for \( i = 1, 2 \). Then
\[
M_1(W) \otimes_{R_2} M_2 \quad \text{and} \quad M_1 \otimes_{R_1} \overline{M_2(W)}
\]
are isomorphic as \( R_1 \)-modules and anti-isomorphic as \( R_2 \)-modules. Both of these relations are realized by the map
\[
M_1 \otimes_{R_1} \mathbb{Z}[K(W)] \otimes_{R_2} M_2 \rightarrow M_1 \otimes_{R_1} \overline{M_2} \otimes_{R_2} \mathbb{Z}[K(W)]
\]
\[
m_1 \otimes k \otimes m_2 \mapsto m_1 \otimes m_2 \otimes \overline{k},
\]
where \( \overline{k} \) is the image of \( k \) under the conjugation automorphism of \( \mathbb{Z}[K(W)] \) induced by the isomorphism \( x \mapsto -x \) of \( K(W) \).
Here we say that $R$-modules $A$ and $B$ are anti-isomorphic if there is an $R$-module isomorphism $A \rightarrow B$. A function $A \rightarrow B$ that gives an $R$-module isomorphism after the identification $B = \overline{B}$ of groups is an anti-isomorphism.

Proof. Note that the map $\mathbb{Z}[K(W)]$ induced by $x \mapsto -x$ on $K(W)$ gives an isomorphism $\mathbb{Z}[K(W)] \rightarrow \overline{\mathbb{Z}[K(W)]}$, where the bar indicates conjugation of both the $R_1$- and $R_2$-module structures. For the purposes of this proof, if $A$ is a module over both $R_1$ and $R_2$ we will indicate by $\overline{A}^{R_1}$ the module $A$ with only the $R_1$-structure conjugated, and similarly with $\overline{A}^{R_2}$ for the $R_2$-structure. In this notation, the claim of the lemma is that the map

$$M_1(W) \otimes_{R_2} \overline{M_2} \rightarrow M_1 \otimes_{R_1} \overline{M_2(W)}^{R_1}$$

given in (10) is an $R_1$-isomorphism and an $R_2$-anti-isomorphism.

By the definition of the $R_1$-module structure on $M_2(W)$, we have

$$\overline{M_2(W)}^{R_1} = M_2 \otimes_{R_2} \mathbb{Z}[K(W)]^{R_1}$$

$$\cong M_2 \otimes_{R_2} \mathbb{Z}[K(W)]^{R_2}$$

$$\cong M_2 \otimes_{R_2} \mathbb{Z}[K(W)],$$

where the second line uses the conjugation isomorphism of the previous paragraph, and the third is an isomorphism of $R_1$-modules (induced by the identity), since $R_2$-conjugation does not affect the $R_1$-structure. Hence as $R_1$-modules,

$$(11) \quad M_1 \otimes_{R_1} \overline{M_2(W)}^{R_1} \cong M_1 \otimes_{R_1} M_2(W),$$

by a map conjugating the $\mathbb{Z}[K(W)]$ factor. The identification

$$M_1 \otimes_{R_1} M_2(W) = M_1(W) \otimes_{R_2} M_2 = M_1(W) \otimes_{R_2} \overline{M_2}$$

of $R_1$-modules is clear (again, since $R_2$-conjugation does not affect the $R_1$-module structure). This proves the first claim of the lemma.

On the other hand, as $R_2$-modules,

$$M_1(W) \otimes_{R_2} \overline{M_2}^{R_2} = M_1 \otimes_{R_1} \overline{\mathbb{Z}[K(W)]}^{R_2} \otimes_{R_2} M_2$$

$$\cong M_1 \otimes_{R_1} \overline{\mathbb{Z}[K(W)]}^{R_1} \otimes_{R_2} M_2$$

$$\cong M_1 \otimes_{R_1} M_2(W)$$

$$\cong M_1 \otimes_{R_1} \overline{M_2(W)}^{R_1},$$

where again in the second line we use conjugation on $\mathbb{Z}[K(W)]$, and the third and fourth lines are identifications of $R_2$-modules. This is the second claim of the lemma. 

We can now give the analogue for twisted coefficients of Theorem 3.5 of [8].

Theorem 3.4 (Duality for twisted coefficients). Let $W : Y_1 \rightarrow Y_2$ be a cobordism and $M_1$ and $M_2$ coefficient modules for $Y_1$ and $Y_2$ as above. Write $W'$ for the manifold $W$ regarded as a cobordism $-Y_2 \rightarrow -Y_1$, and let $s$ be a spin$^c$ structure on $W$ with restrictions $s_1 = s|_{Y_1}$. Then for any $\xi \in HF^+(Y_1, s_1; M_1)$ and $\eta \in HF^-(Y_2, s_2; M_2)$, we have

$$\langle F^+_W, s(\xi), \eta \rangle = \langle \xi, F^-_{W', s}(\eta) \rangle,$$

under the identification of coefficient modules given in Lemma 3.3.
Proof. We adapt the proof from [8]. Decompose $W$ into a composition of 1-handle additions, followed by 2-handles and then 3-handles. The verification of duality for 1- and 3-handle cobordisms is unchanged from the untwisted case given in [8], so we omit it here.

Assume, then, that $W$ is a cobordism comprised entirely of 2-handle additions. Let $R$ denote the reflection of the standard 2-simplex $\Delta$ that fixes one corner and exchanges the other two. Specifically, if the edges are labeled $e_\alpha$, $e_\beta$ and $e_\gamma$, we take $R$ to exchange $e_\beta$ and $e_\gamma$ while reversing $e_\alpha$. If $A_W$ is an additive assignment for a Heegaard triple $(\Sigma, \alpha, \beta, \gamma, z)$ associated to $W$ as in Definition 2.5 (using a base triangle $\psi_0$), then we obtain an additive assignment $A_{W'}$ for $W'$ from the triangle $\psi_0 \circ R$.

More generally, for any (homotopy class of) triangle $\psi \in \pi_2(x, y, w)$, precomposition with $R$ gives a triangle $\psi \circ R \in \pi_2(w, y, x)$. Moreover, if $\psi = \psi_0 + \phi_{\alpha\beta} + \phi_{\beta\gamma} + \phi_{\alpha\gamma}$ then it is easy to see that

$$
\psi \circ R = \psi_0 \circ R + (\phi_{\alpha\beta} \circ r) + \phi_{\beta\gamma} + (\phi_{\alpha\gamma} \circ r),
$$

where $r$ is the reflection across the real axis used previously. Therefore

$$
A_{W'}(\psi \circ R) = \delta(A_1(\phi_{\alpha\beta} \circ r) + A_2(\phi_{\alpha\gamma} \circ r)) = -A_W(\psi)
$$

(c.f. the proof of Lemma 3.2). Furthermore, just as in the case of disks we have an identification

$$
M_{-\Sigma}(\psi \circ R) = M_\Sigma(\psi).
$$

Thus for $m_i \in M_i$:

$$
\langle F_{W,s}(\langle x, i \rangle m_1), [w, k]m_2 \rangle = \left\langle \sum_{\psi \in \pi_2(x, \theta, v)} \#M_{\Sigma}(\psi) \cdot [v, i - n_z(\psi)]m_1 \otimes e^{A_W(\psi)}, [w, k]m_2 \right\rangle
$$

$$
= \sum_{\psi \in \pi_2(x, \theta, w)} \#M_{\Sigma}(\psi) \cdot (m_1 \otimes e^{A_W(\psi)}) \otimes m_2,
$$

an element of $M_1(W) \otimes \overline{M}_2$. On the other hand,

$$
\langle [x, i]m_1, F_{W',s}(\langle w, k \rangle m_2) \rangle = \left\langle [x, i]m_1, \sum_{\tilde{\psi} \in \pi_2(w, \theta, v)} \#M_{-\Sigma}(\tilde{\psi}) \cdot [v, i - n_z(\tilde{\psi})]m_2 \otimes e^{A_{W'}(\tilde{\psi})} \right\rangle
$$

$$
= \sum_{\tilde{\psi} \in \pi_2(w, \theta, x)} \#M_{-\Sigma}(\tilde{\psi}) \cdot m_1 \otimes (m_2 \otimes e^{A_{W'}(\tilde{\psi})})
$$

$$
= \sum_{\psi \in \pi_2(x, \theta, w)} \#M_{-\Sigma}(\psi \circ R) \cdot m_1 \otimes (m_2 \otimes e^{A_{W'}(\psi \circ R)})
$$

$$
= \sum_{\psi \in \pi_2(x, \theta, w)} \#M_{\Sigma}(\psi) \cdot m_1 \otimes (m_2 \otimes e^{-A_W(\psi)})
$$

in $M_1 \otimes \overline{M}_2(W)^{R_1}$. This is the image of the previous result under our identification $M_1(W) \otimes \overline{M}_2 \cong M_1 \otimes \overline{M}_2(W)^{R_1}$, which proves the theorem. \qed
4. Action of First Homology

In this section we extend to twisted coefficients an additional aspect of the algebraic structure of Heegaard Floer homology, namely the action of $\Lambda^*(H_1(Y;\mathbb{Z})/\text{tors})$ on $HF^0(Y,s)$. We also discuss the interaction of this structure with cobordism-induced homomorphisms. Much of this section is a straightforward generalization of material from [6, 7, 8], so we omit many of the details.

**Proposition 4.1.** Fix an oriented spin$^c$ 3-manifold $(Y,s)$ and a module $M$ for $R_Y = \mathbb{Z}[H^1(Y)]$. Then for any $h \in H_1(Y;\mathbb{Z})/\text{tors}$ there is a chain endomorphism $A_h$ of $CF^\infty(Y,s;M)$ of degree $-1$, equivariant with respect to $U$ and the $R_Y$ action, with the property that $A_h \circ A_h$ is chain homotopic to $0$.

Thus, the collection of maps $A_h$ provides $HF^0(Y,s;M)$ with the structure of a module over $R_Y[U] \otimes \Lambda^*(H_1(Y;\mathbb{Z})/\text{tors})$.

**Proof.** For a generator $[x,i] \otimes m \in CF^\infty(Y,s;M)$ we set

$$A_h([x,i] \otimes m) = \sum_{y \in T_\alpha \cap T_\beta} \sum_{\phi \in \pi_2(x,y) \atop \mu(\phi) = 1} \#\hat{M}(\phi) \langle A(\phi), h \rangle \cdot [y,i-n_x(\phi)] \otimes e^{A(\phi)} \cdot m.$$ 

(Here $A(\phi)$ denotes the value of the additive assignment used in the definition of twisted Floer homology.) Then the proof that $A_h$ is a chain map whose square is trivial in homology is virtually identical to the proof in the untwisted case (c.f. Proposition 4.17 of [6]).

We will omit the map $A_h$ from the notation and simply write $h._x\xi$ for the action of $h$ on the element $\xi \in HF^0(Y,s;M)$.

**Remark 4.2.** Though the action of $H_1(Y)/\text{tors}$ is defined for Floer homology with any coefficients, it may be largely trivial. Indeed, suppose $M$ is an $R_Y$-module, and let $Z_M \subset H^1(Y)$ denote the stabilizer of $M$ (c.f. Definition 2.10). Then it can be shown that if $h \in H_1(Y)$ has the property that

$$\langle \alpha, h \rangle = 0 \quad \text{for all } \alpha \in Z$$

then $A_h$ is chain homotopic to $0$. In particular, this implies that the $H_1(Y)/\text{tors}$ action on the fully twisted homology $HF^0(Y,s;R_Y)$ is trivial.

**Lemma 4.3.** Let $(Y,s)$ be as above, and let $M$ and $N$ be modules for $R_Y$. Then for any $h \in H_1(Y)/\text{tors}$, any $\xi \in HF^+(Y,s;M)$ and any $\eta \in HF^-(Y,s;N)$ we have

$$\langle h._x\xi, \eta \rangle = -\langle \xi, h._x\eta \rangle.$$ 

**Proof.** This follows from a calculation very similar to the one in Lemma 3.2. Indeed, the only difference is the appearance of the factors $\langle A(\phi), h \rangle$, which change sign under orientation reversal.

We now extend the twisted cobordism invariants from the previous section to include the action of first homology. Specifically, for a cobordism $W : Y_0 \to Y_1$ we wish to define $F^0_{W,s}$ as a map

$$F^0_{W,s} : HF^0(Y_0,s_0;M) \otimes \Lambda^*H_1(W)/\text{tors} \to HF^0(Y_1,s_1;M(W)).$$
With the preceding in place the definition runs precisely as in the untwisted case; we summarize the construction.

Suppose first that $W : Y_0 \to Y_1$ is a cobordism consisting only of 2-handle additions. Then $H_1(W, \partial W) = 0$, so the map

$$i_* = i_{0*} - i_{1*} : H_1(Y_0)/\text{tors} \oplus H_1(Y_1)/\text{tors} \to H_1(W)/\text{tors}$$

is surjective. Fix $h \in H_1(W)/\text{tors}$ and suppose $h = i_*(h_0, h_1)$. For $\xi \in HF^*(Y_0, s_0; M)$, we set

$$F_{W,s}^0(\xi \otimes h) = F_{W,s}^0(h_0, \xi) - h_1 F_{W,s}^0(\xi).$$

Clearly $F_{W,s}^0((\xi \otimes h) \otimes h) = 0$, so the action extends to $\Lambda^*H_1(W)/\text{tors}$.

In fact, we can define this action using a Heegaard triple $(\Sigma, \alpha, \beta, \gamma, z)$ describing the cobordism, just as in Lemma 2.6 of [8]. It follows as in that proof that the action of pairs $(h_0, h_1)$ in the image of $H_2(W, \partial W, \mathbb{Z})$ is trivial, so the action descends as claimed to $H_1(W)/\text{tors}$.

In general for a cobordism containing 1-, 2-, and 3-handles we write the induced homomorphism as a composition $F_W^0 = E^0 \circ H^\circ \circ G^\circ$ as in section 2.2. This composition corresponds to a factorization $W = W_3 \cup W_2 \cup W_1$ where $W_i$ includes only handles of index $i$. As observed in [8], the inclusion induces an isomorphism $H_1(W_2) \to H_1(W)$; thus for $\omega \in \Lambda^*H_1(W)/\text{tors}$ we set

$$F_W^0(\xi \otimes h) = E^0(H^\circ (G^\circ (\xi) \otimes \omega))$$

just as in [8].

Many properties of the extended cobordism maps (12) follow from corresponding properties of the original ones. We mention two results here.

**Theorem 4.4.** Let $W : Y_0 \to Y_1$ be a cobordism with spin$^c$ structure $s$ and suppose $\omega \in \Lambda^*H_1(W)/\text{tors}$. Write $s_i$ for $s|_{Y_i}$. Then for modules $M$ and $N$ over $R_{Y_0}$ and $R_{Y_1}$ respectively, and for any $\xi \in HF^+(Y_0, s_0; M)$ and $\eta \in HF^-(Y_1, s_1, N)$, we have

$$\langle F_{W,s}^+(\xi \otimes \omega), \eta \rangle = \langle \xi, F_{W,s}^-(\eta \otimes \omega) \rangle.$$

**Proof.** Assume first that $W$ consists of 2-handles only, and suppose $h \in H_1(W)/\text{tors}$ has the expression $h = i_*(h_0, h_1)$ for $h_i \in H_1(Y_i)/\text{tors}$. Then using the duality theorem for twisted coefficients (Theorem 3.4) and Lemma 4.3 we have

$$\langle F_{W,s}^+(\xi \otimes h), \eta \rangle = \langle F_{W,s}^+(h_0, \xi) - h_1 F_{W,s}^+(\xi), \eta \rangle = -\langle \xi, h_0 F_{W,s}^-(\eta) \rangle + \langle \xi, F_{W,s}^+(h_1, \eta) \rangle = \langle \xi, F_{W,s}^-(\eta \otimes h) \rangle.$$

It is a simple matter to extend to general cobordisms and general $\omega$. \hfill $\square$

**Theorem 4.5.** The composition law (Theorem 2.9) holds for the extended maps (12). More precisely, suppose $W = W_1 \cup Y_1 W_2$ is a composite cobordism and write

$$j_* : \Lambda^*(H_1(W_1)/\text{tors}) \otimes \Lambda^*(H_1(W_2)/\text{tors}) \to \Lambda^*(H_1(W)/\text{tors})$$

for the surjection induced on exterior algebras by the Mayer-Vietoris map $H_1(W_1) \oplus H_1(W_2) \to H_1(W)$. Fix $\omega_i \in \Lambda^*H_1(W_i)/\text{tors}$ and write $\omega$ for the image of $\omega_1 \otimes \omega_2$ under $j_*$. Then we
can find choices of representatives for the maps $F^\circ$ such that for any spin$^c$ structure $s$ on $W$ and any $\alpha \in H^1(Y_1)$, we have

$$F^\circ_{W,s+\delta\alpha}(\xi \otimes \omega) = \Pi_W \left[ F^\circ_{W_2,s|W_2}(\epsilon^\alpha \cdot F^\circ_{W_1,s|W_1}(\xi \otimes \omega_1) \otimes \omega_2) \right].$$

**Proof.** This follows from Theorem 2.9 together with the formal properties of the $H_1$-action, particularly (13) in the case of 2-handles. (See [8], particularly Proposition 4.20. Note that here the strengthened composition law means that summing over spin$^c$ structures is unnecessary.) \qed

5. INVARIANTS FOR 4-MANIFOLDS

We briefly recall the definition of Ozsváth-Szabó 4-manifold invariants from [8], and then proceed to discuss their calculation in the context of 4-manifolds obtained by gluing two manifolds with boundary.

Suppose $X$ is a closed 4-manifold having $b^+(X) \geq 2$. Then we can find an *admissible cut* for $X$: that is, a hypersurface $N \subset X$ separating $X$ into components $X = V_1 \cup_N V_2$ with the following properties:

1. For $i = 1, 2$, we have $b^+(V_i) \geq 1$.
2. The image of the Mayer-Vietoris map $\delta : H^1(N) \to H^2(X)$ is trivial.

As observed previously (Remark 2.8), the second condition ensures that spin$^c$ structures on $X$ are determined by their restrictions to $V_1$ and $V_2$.

The first condition is relevant because of the following.

**Lemma 5.1** ([8]). If $W$ is a cobordism having $b^+(W) \geq 1$ then for any spin$^c$ structure $s$ and in any coefficient module, the map $F^\infty_{W,s}$ vanishes.

Recall that for all sufficiently large integers $r$, the subgroups $\ker(U^r_+) \subset HF^-(Y, s)$ and $\text{Im}(U^r_-) \subset HF^+(Y, s)$ are independent of $r$ (where $U^r_+$ denotes the action of $U$ on $HF^+$). The *reduced Floer homology groups* are defined by $HF^-_{\text{red}}(Y, s) = \ker(U_+^r)$ and $HF^+_{\text{red}}(Y, s) = \text{coker}(U_+^r)$. From the long exact sequence

$$\cdots \to HF^\infty(Y, s) \to HF^+(Y, s) \to \tau HF^-(Y, s) \to \cdots$$

and the fact that $U$ is an isomorphism on $HF^\infty$ we see that Lemma 5.1 implies that the image of $F^\infty_{W,s}$ for $W$ a cobordism with $b^+(W) \geq 1$ lies in $HF^\infty_{\text{red}}$, while $F^+_{W,s}$ factors through $HF^+_{\text{red}}$. Note also that the homomorphism $\tau$ in the sequence induces an isomorphism

$$\tau : HF^+_{\text{red}}(Y, s) \to HF^-_{\text{red}}(Y, s).$$

(All of the above holds in any coefficient system).

**Definition 5.2** ([8]). Let $\Theta^-$ denote a top-degree generator of $HF^-(S^3)$. Let $N$ be an admissible cut for a 4-manifold $X$ as above, and fix a spin$^c$ structure $s$ on $X$. The Ozsváth-Szabó invariant of $(X, s)$ is the integer-valued function

$$\Phi_{X,s} : \mathbb{A}(X) := \mathbb{Z}[U] \otimes \Lambda^*(H_1(X)/\text{tors}) \to \mathbb{Z}$$

defined by

$$\Phi_{X,s}(U^n \otimes \omega) = \langle (F^+_{V_2} \circ \tau^{-1} \circ F^-_{V_1})(U^n \cdot \Theta^- \otimes \omega), \Theta^- \rangle.$$
Remark 5.3. \( \Phi_{X,s} \) is defined only modulo a sign, due to the sign ambiguity of the maps associated to cobordisms.

Remark 5.4. It follows from the formula for the degree shift induced by a cobordism that \( \Phi_{X,s} \) is nonzero only on elements of \( \mathbb{A}(X) \) having degree \( d(s) \), where

\[
d(s) = \frac{1}{4}(c_2(s) - 2e(X) - 3\sigma(X)).
\]

Here \( e(X) \) is the Euler characteristic of \( X \) and \( \sigma(X) \) is the signature, and \( \mathbb{A}(X) \) is graded so that \( U \) carries degree 2 and elements of \( H_1(X)/\text{tors} \) carry degree 1.

Remark 5.5. In \cite{Ozsvath2003}, Ozsváth and Szabó extend the definition of \( \Phi \) to cover certain situations in which \( b^+(X) = 1 \). This extension depends on the choice of a line in \( H_2(X; \mathbb{Q}) \) spanned by an element \( v \) of square 0, and a spin\(^c\) structure \( s \) for which \( \langle c_1(s), v \rangle \neq 0 \). Suppose \( N \) is a cut for \( X \) with the property that the image of \( H_2(N; \mathbb{Q}) \) lies in \( L \). In this situation one can define a projection \( HF^-(N,s) \rightarrow HF^\text{red}(N,s) \) and by incorporating this projection in the composition used to define \( \Phi \), make sense of the 4-manifold invariant. We will not deal directly with this construction until the next section; for now it suffices to observe that the condition on \( b^+ \) is imposed mainly to take advantage of Lemma 5.1: it will typically suffice for our purposes to show that the conclusion of that lemma holds in the relevant circumstances.

Ozsváth and Szabó show that \( \Phi_{X,s} \) does not depend on the choice of admissible cut \( N \), and therefore gives an invariant of smooth spin\(^c\) 4-manifolds with \( b^+ \geq 2 \). An important property of \( \Phi_{X,s} \) is that it is nonzero for at most finitely many spin\(^c\) structures \( s \) on \( X \).

In many situations there are convenient decompositions \( X = Z_1 \cup_Y Z_2 \), in which \( Y \) fails to be admissible in the sense above—specifically, condition (2) in the definition of admissibility is violated. Ozsváth and Szabó prove that one can use such a cut to obtain information about sums of invariants of \( X \) (Lemma 8.8 of \cite{Ozsvath2002}), but in order to obtain more detailed information we must pass to twisted coefficients.

We express our results in terms of group rings. In the situation of cutting \( X \) along a 3-manifold \( Y \) satisfying (1) but not (2) in the definition of admissible cut, the relevant group is \( K(X,Y) = \ker(H^2(X) \rightarrow H^2(Z_1) \oplus H^2(Z_2)) \) (c.f. Remark 2.8). For a given \( s \in Spin^c(X) \) and \( \alpha \in \mathbb{A}(X) \), we would like a way to calculate the element

\[
\sum_{t \in K(X,Y)} \Phi_{X,s+t}(\alpha) \cdot e^t \in \mathbb{Z}[K(X,Y)]
\]

in terms of invariants on the manifolds-with-boundary \( Z_1 \) and \( Z_2 \). Indeed, the invariants of all spin\(^c\) structures on \( X \) can be read from the coefficients of the above expressions for various \( s \).

As a slight abuse of notation, if \( Z \) is a 4-manifold with one boundary component \( Y \), and \( s \) is a spin\(^c\) structure on \( Z \), we will denote by \( F^\infty_{Z,s} \) the homomorphism \( HF^\infty(S^3) \rightarrow HF^\infty(Y) \) induced by the cobordism obtained by removing a 4-ball from the interior of \( Z \). Since we only need to refer to maps in both twisted and untwisted Floer homology, we will follow the notation of Ozsváth and Szabó and write \( \overline{F}^\infty_W \) for the map in twisted coefficients induced by \( W \) and \( F^\infty_W \) for the untwisted map.
Definition 5.6. Suppose $Z$ is an oriented 4-manifold with connected boundary $\partial Z = Y$ and $s \in \text{Spin}^c(Z)$. Define the relative Ozsváth-Szabó invariant $\Psi_{Z,s}$ of $Z$ to be the function

$$\Psi_{Z,s} : \mathbb{A}(Z) \rightarrow HF^-_{\text{red}}(Y,s|_Y; Z[K(Z)])$$

given by

$$\Psi_{Z,s}(U^n \otimes \omega) = [E_{Z,s}(U^n \cdot \Theta^{-} \otimes \omega)].$$

Here the brackets indicate equivalence class under the action of $K(Z)$, where $K(Z) = \ker(H^2(Z,Y) \to H^2(Z)).$

Recall that the twisted-coefficient map $E_{Z,s}$ is defined only up to the action of $\delta(H^1(\partial Z)) = K(Z)$. Note also that if $b^+(Z) \geq 1$ then $\Psi_{Z,s}$ takes values in $HF^-_{\text{red}}(Y)$.

The following result is a refinement of Theorem 1.6 from the introduction, and shows how to calculate (14) in terms of relative invariants.

**Theorem 5.7.** Let $X$ be a closed 4-manifold with $b^+(X) \geq 2$ and $Y \subset X$ a 3-dimensional submanifold separating $X$ into components $Z_1$ and $Z_2$. Let $s$ be a spin$^c$ structure on $X$ and write $s_i = s|_{Z_i}$. Assume that $\Psi_{Z_i,s_i}$ takes values in $HF^-_{\text{red}}$ for $i = 1, 2$, and also that $b^+(Z_i) \geq 1$ for at least one of $Z_1, Z_2$. Then for any $\alpha_i \in \mathbb{A}(Z_i)$ we have

$$\sum_{i \in K(X,Y)} \Phi_{X,s-i}(\alpha) \cdot e^i = \langle \tau^{-1}(\Psi_{Z_1,s_1}(\alpha_1)), \Psi_{Z_2,s_2}(\alpha_2) \rangle$$

as elements of $\mathbb{Z}[K(X,Y)]$, up to sign and multiplication by an element of $K(X,Y)$. Here $\alpha$ is the image of $\alpha_1 \otimes \alpha_2$ under the natural map $\mathbb{A}(Z_1) \otimes \mathbb{A}(Z_2) \to \mathbb{A}(X)$.

In the statement of the theorem, we are implicitly choosing representatives for $\Psi_{Z_i,s_i}(\alpha_i)$ and pairing them using the twisted-coefficient pairing defined earlier. Lemma 2.7 shows that the pairing does indeed take values in $\mathbb{Z}[K(X,Y)]$, and it follows also that different choices of representatives give rise to elements of $\mathbb{Z}[K(X,Y)]$ differing by multiplication by an element of $K(X,Y)$.

The rest of this section is devoted to the proof of Theorem 5.7. For simplicity, we focus on the case $\alpha = 1$ in the following; the general case follows by an entirely analogous argument with Theorems 4.4 and 4.5 replacing Theorems 3.4 and 2.9.

We begin with a few easy preparatory lemmas.

**Lemma 5.8.** Fix a spin$^c$ 3-manifold $Y$ and $R_Y$-module $M$ and $N$. Let $\phi : M \to N$ be a module homomorphism, and write $\phi_* : HF^0(Y; M) \to HF^0(Y; N)$ for the induced map in Floer homology. Then the following diagram commutes:

$$\begin{array}{ccc}
HF^{-}(Y; M) & \to & HF_{\infty}(Y; M) & \to & HF^{+}(Y; M) \\
\phi_* \downarrow & & \phi_* \downarrow & & \phi_* \downarrow \\
HF^{-}(Y; N) & \to & HF_{\infty}(Y; N) & \to & HF^{+}(Y; N)
\end{array}$$

In particular, $\phi_*$ descends to a map on reduced homology, and commutes with $\tau$ (and $\tau^{-1}$).

**Proof.** This is clear. $\square$
Lemma 5.9. For \( i = 1, 2 \) let \( M_i \) and \( N_i \) be \( R_Y \)-modules, and consider homomorphisms \( \phi : M_1 \to N_1 \) and \( \psi : M_2 \to N_2 \). For any \( \xi \in HF^+(Y; M_1) \) and \( \eta \in HF^-(Y; M_2) \), we have

\[
(\phi_*(\xi), \psi_*(\eta)) = \phi \otimes \psi((\xi, \eta)) \in N_1 \otimes_{R_Y} N_2
\]

Proof. This follows easily from the definitions. \(\square\)

Lemma 5.10. Suppose \( W = W_1 \cup Y \cup W_2 : Y_0 \to Y_2 \) is a composite cobordism, and \( s_1 \) and \( s_2 \) are spin\(^c\) structures on \( W_1 \) and \( W_2 \) with \( s_1|_{Y_1} = s_2|_{Y_1} \). If \( F_{W_1, s_1}^- \) has image in \( HF_{red}^-(Y_1) \) then for any coefficient module \( M \) for \( Y_0 \),

1. \( \text{Im}(F_{W_2, s_2}^- \circ F_{W_1, s_1}^-) \subset HF_{red}^-(Y_2; M(W_1)(W_2)) \), and
2. \( \tau^{-1} \circ F_{W_2, s_2}^- \circ F_{W_1, s_1}^- = F_{W_2, s_2}^- \circ \tau^{-1} \circ F_{W_1, s_1}^- \).

Proof. (1) is clear from the fact that \( F_{W_2, s_2}^- \) maps \( HF_{red}^-(Y_1; M(W_1)) \) into \( HF_{red}^-(Y_2; M(W_1)(W_2)) \).

Statement (2) follows from the commutative diagram

\[
\begin{array}{ccc}
HF^+(Y_1; M(W_1)) & \xrightarrow{F_{W_2, s_2}^+} & HF^+(Y_2; M(W_1)(W_2)) \\
\downarrow \tau & & \downarrow \tau \\
HF^-(Y_0; M) & \xrightarrow{F_{W_1, s_1}^-} & HF^-(Y_1; M(W_1)) & \xrightarrow{F_{W_2, s_2}^-} & HF^-(Y_2; M(W_1)(W_2))
\end{array}
\]

together with part (1). \(\square\)

With these preliminaries in place, we turn our attention to the proof of Theorem 5.7 (with \( \alpha_1 = \alpha_2 = \alpha = 1 \)). Thus, let \( X = Z_1 \cup_Y Z_2 \) be as in the statement of the theorem, and let us assume \( b^+(Z_2) \geq 1 \). Then we can find an admissible cut \( N \) for \( X \) contained in \( Z_2 \) (c.f. the construction in example 8.4 of [8]). Suppose \( X \) is decomposed into pieces \( V_1 \) and \( V_2 \) along \( N \), so that

\[
X = V_1 \cup_N V_2 = Z_1 \cup_Y W \cup_N V_2
\]

where \( W = V_1 \cap Z_2 \) is a cobordism \( Y \to N \).

Let us fix a spin\(^c\) structure \( s \) on \( X \). For simplicity we will omit the spin\(^c\) structure from the notation for homomorphisms induced by cobordisms, but all relevant cobordisms and their boundaries will be equipped with spin\(^c\) structures obtained by restricting \( s \).

By definition, we have

\[
\Phi_{X, s}(1) = \langle F_{V_1}^+ \circ \tau^{-1} \circ F_{V_1}^-(\Theta^-), \Theta^- \rangle
\]

\[
= \langle \tau^{-1} \circ F_{V_1}^-(\Theta^-), F_{V_2}^-(\Theta^-) \rangle = \langle \tau^{-1} \circ \epsilon_* \circ F_{V_1}^-(\Theta^-), \epsilon_* \circ F_{V_2}^-(\Theta^-) \rangle
\]

(16)

\[
= \langle \tau^{-1} \circ E_{V_1}^- (\Theta^-), E_{V_2}^- (\Theta^-) \rangle
\]

We have passed to twisted coefficients using the remark after Theorem 2.9. The last line uses Lemma 5.8 and the twisted pairing which takes values in \( \mathbb{Z}[K(X, N)] \). Since \( N \) is admissible the group \( K(X, N) \) is trivial and hence the pairing is \( \mathbb{Z} \)-valued; the homomorphism \( \epsilon_* \otimes \epsilon_* \) arising from Lemma 5.9 is the identity here.

According to Theorem 2.9 we can find representatives for the maps involved that satisfy

\[
\tilde{F}_{V_1}^- = \Pi_{V_1} \circ F_{W}^+ \circ \tilde{F}_{Z_1}^-,
\]
where $\Pi_{V_1}$ is the map induced in homology by a projection map $\mathbb{Z}[K(V_1, Y)] \to \mathbb{Z}[K(V_1)]$, which we also denote by $\Pi_{V_1}$. Different choices of representatives for $[E_{V_1}]$ and the other maps differ by the action of $R_N$ on $\mathbb{Z}[K(X, N)] = \mathbb{Z}$, which is trivial. Hence we can replace (16) with

$$\Phi_{X, s}(1) = \langle \tau^{-1} \circ \Pi_{V_1} \circ E_{W}^{-1} \circ E_{Z_1}^{-1}(\Theta^-), E_{V_2}^{-1}(\Theta^-) \rangle$$

(17)

$$= \Pi_{V_1} \otimes 1 \cdot \langle \tau^{-1} \circ E_{W}^{-1} \circ E_{Z_1}^{-1}(\Theta^-), E_{V_2}^{-1}(\Theta^-) \rangle$$

\textbf{Lemma 5.11.} Under the isomorphism

$$\mathbb{Z}[K(V_1, Y)] \otimes_{R_N} \mathbb{Z}[K(V_2)] \cong \mathbb{Z} \left[ \frac{K(V_1, Y) \oplus K(V_2)}{H^1(N)} \right],$$

the map $\Pi_{V_1} \otimes 1$ corresponds to the homomorphism $\Pi_Z$ sending an element of a group ring to the coefficient of the identity element.

\textbf{Proof.} We have a diagram of identifications

$$\begin{array}{ccc}
\mathbb{Z}[K(V_1, Y)] \otimes_{R_N} \mathbb{Z}[K(V_2)] & \xrightarrow{\Pi_{V_1} \otimes 1} & \mathbb{Z}[K(V_1)] \otimes_{R_N} \mathbb{Z}[K(V_2)] \\
\downarrow & & \downarrow \\
\mathbb{Z} \left[ \frac{K(V_1, Y) \oplus K(V_2)}{H^1(N)} \right] & \xrightarrow{p} & \mathbb{Z} \left[ \frac{K(V_1) \oplus K(V_2)}{H^1(N)} \right]
\end{array}$$

Again, since $N$ is admissible

$$\frac{K(V_1) \oplus K(V_2)}{H^1(N)} = \ker(H^2(X) \to H^2(V_1) \oplus H^2(V_2)) = 0.$$  

The projection $p$ is induced by some map

$$\frac{K(V_1) \oplus K(V_2)}{H^1(N)} \to \frac{K(V_1, Y) \oplus K(V_2)}{H^1(N)},$$

for which there is only one choice since the domain group is trivial. The construction of $p$ from this map proves the claim. \qed

Returning with this to equation (17), we have

$$\Phi_{X, s}(1) = \Pi_Z(\tau^{-1} \circ \Pi_{V_1} \circ E_{W}^{-1} \circ E_{Z_1}^{-1}(\Theta^-), E_{V_2}^{-1}(\Theta^-))$$

$$= \Pi_Z(\tau^{-1} \circ E_{W}^{-1} \circ E_{Z_1}^{-1}(\Theta^-), E_{V_2}^{-1}(\Theta^-))$$

(18)

using Lemma 5.10, Theorem 3.4 and the identification

$$\mathbb{Z}[K(V_1, Y)] \otimes_{R_N} \mathbb{Z}[K(V_2)] \cong \mathbb{Z}[K(Z_1)] \otimes_{R_Y} \mathbb{Z}[K(Z_2, N)]$$

provided by Lemma 3.3. Note that both these modules can be written

$$\mathbb{Z} \left[ \frac{K(Z_1) \oplus K(W) \oplus K(V_2)}{H^1(Y) \oplus H^1(N)} \right].$$
We would like to apply the composition law in (18) to replace $F_W^- \circ F_{V_2}^-$ by $F_{Z_2}^-$, but we are missing a factor of $\Pi_{Z_2}$ required by Theorem 2.9. By commutativity of the square (*) in the following diagram, we are free to introduce this factor:

Indeed, it follows that $\Pi_Z = \Pi_Z \circ (1 \otimes \Pi_{Z_2})$ (after identifying the groups in the column on the left). Thus (18) becomes:

$$\Phi_{X,s}(1) = \Pi_Z \circ (1 \otimes \Pi_{Z_2}) \cdot \langle \tau^{-1} \circ F_{Z_1}^- (\Theta^-), F_W^- \circ F_{V_2}^-(\Theta^-) \rangle$$

After possibly translating by an element of $R_Y$. This verifies the “constant coefficient” of (15). For the general statement, suppose $t = \delta h \in K(X,Y)$. Then since $s - t = s$ when restricted to $V_2$ we can follow the same steps as above (and using the second part of Theorem 2.9) to see

$$\Phi_{X,s-t}(1) = \langle \tau^{-1} \circ F_{V_1,s-t}^- (\Theta^-), F_{V_2,s}^- (\Theta^-) \rangle$$

where we can use the same representatives for $[F_{Z,s}]$ as before. Since the action of $R_Y$ on $\mathbb{Z}[K(X,Y)]$ is via the coboundary, this last expression is exactly the coefficient of $e^{\delta h} = e^t$ in $\langle \tau^{-1} \circ F_{Z_1,s}^- (\Theta^-), F_{Z_2,s}^- (\Theta^-) \rangle$. This completes the proof of Theorem 5.7.

6. Product Formulae

The theorems of the introduction follow from Theorem 5.7 together with results of the authors [2] on the Heegaard Floer homology of 3-manifolds of the form $Y = \Sigma_g \times S^1$, where $\Sigma_g$ is a Riemann surface of genus $g$. 
6.1. Algebraic Preliminaries. We begin by recalling some notation: for nonnegative integers \( g \) and \( d \), we set
\[
X(g, d) = \bigoplus_{i=0}^{d} \Lambda^{2g-i}H^1(\Sigma_g) \otimes \mathbb{Z}[U]/U^{d-i+1}.
\]
It will be convenient to regard \( X(g, d) \) as a subgroup of \( \Lambda^*H^1(\Sigma_g) \otimes \mathbb{Z}[U] \). Furthermore, we consider the latter group as graded such that the summand \( \Lambda^{2g-i}H^1(\Sigma_g) \otimes U^n \) is homogeneous of degree \( g-i-2n \). Note that in fact there is an isomorphism \( X(g, d) \cong H^*(\text{Sym}^d(\Sigma_g)) \) of (relatively) graded groups (see [3]).

We define an action of \( \Lambda^*H_1(\Sigma_g) \) on \( X(g, d) \), denoted by \( \cap \), as follows: for \( h \in H_1(\Sigma_g) \) and \( \omega \otimes U^j \) a homogeneous element of \( \Lambda^*H^1(\Sigma_g) \otimes \mathbb{Z}[U] \), we let
\[
(19) \quad h \cap (\omega \otimes U^j) = i_h \omega \otimes U^j + PD(h) \wedge \omega \otimes U^{j+1}.
\]
The action on \( X(g, d) \) is the obvious induced action.

**Theorem 6.1.** Let \( \mathfrak{s} \) be a spin\(^c\) structure on \( Y = \Sigma_g \times S^1 \), \( g \geq 1 \). If \( c_1(\mathfrak{s}) \) is not of the form \( 2kPD[S^1] \) for some integer \( k \) with \( |k| \leq g-1 \) then \( HF^+(\Sigma_g \times S^1, \mathfrak{s}; M) = 0 \) for any coefficient module \( M \).

Suppose \( c_1(\mathfrak{s}) = 2kPD[S^1] \) where \( 0 < |k| \leq g-1 \). Then there is an isomorphism of groups
\[
HF^+(\Sigma_g \times S^1, \mathfrak{s}; \mathbb{Z}) \cong X(g, d),
\]
where \( d = g-1-|k| \). Under this identification, for any \( h \in H_1(\Sigma_g) \) and \( \xi \in HF^+(\Sigma_g \times S^1, \mathfrak{s}; \mathbb{Z}) \), we have
\[
h \cdot \xi = h \cap \xi + \text{terms of lower degree},
\]
where “lower degree” refers to a certain lift of the relative cyclic grading on \( HF^+(\Sigma_g \times S^1) \) to a relative \( \mathbb{Z} \) grading.

More generally, if \( L(t) \) denotes the ring of integer Laurent polynomials in a single variable \( t \), made into a \( \mathbb{Z}[H^1(\Sigma_g \times S^1)] \)-module as in Example 2.12, then there is an isomorphism
\[
HF^+(\Sigma_g \times S^1, \mathfrak{s}; L(t)) \cong X(g, d) \otimes_{\mathbb{Z}} L(t)
\]
of \( L(t) \)-modules. The action of \( H_1(\Sigma_g) \) on \( HF^+(\Sigma_g \times S^1; L(t)) \) is given by the cap product to top order, as in the untwisted case.

Implicit in these statements is the fact that \( [S^1] \in H_1(\Sigma_g \times S^1) \) acts trivially on the relevant Floer homology groups.

**Proof.** The statement with \( \mathbb{Z} \) coefficients was proved by Ozsváth and Szabó in [10]. For the group structure in the case of \( L(t) \) coefficients, see [2]. The statement regarding the action of \( H_1(\Sigma_g) \) follows in the same way as in the untwisted case: the proof given in [10] works here since the groups with \( L(t) \) coefficients are trivially twisted.

We will have occasion below to consider the cobordism from \( S^3 \) to \( \Sigma_g \times S^1 \) obtained by removing a 4-ball from \( \Sigma_g \times D^2 \). The spin\(^c\) structures on this cobordism are in natural correspondence with the integers, by declaring that \( \mathfrak{s}_k \) is the spin\(^c\) structure with \( \langle c_1(\mathfrak{s}_k), [\Sigma_g] \rangle = 2k \). We will write \( \mathfrak{s}_k \) also for the restriction of \( \mathfrak{s}_k \) to \( \Sigma_g \times S^1 \). Note that taking \( \mathbb{Z} \) coefficients for the Heegaard Floer homology of \( S^3 \), this cobordism induces the coefficient module \( L(t) \) on \( \Sigma_g \times S^1 \).
Observe that when $k \neq 0$ we have $HF^+(\Sigma_g \times S^1, s_k; L(t)) = HF^\text{red} (\Sigma_g \times S^1, s_k; L(t))$, which follows since a sufficiently high power of $U$ annihilates $HF^+(\Sigma_g \times S^1, s_k; L(t))$. In general, we have an identification
\[ \tau : HF^\text{red}_+ (Y, s; M) \cong HF^\text{red}_- (Y, s; M) \]
of $\Lambda^* H_1(Y) \otimes \mathbb{Z}[U]$-modules, so combining these facts with Theorem 6.1 shows that $HF^\text{red}_- (\Sigma_g \times S^1, s_k; L(t)) \cong X(g, d) \otimes L(t)$ with action by $H_1(\Sigma_g)$ given to leading order by the cap product (19).

We can define an $L(t)$-valued form $(\cdot, \cdot)$ on $HF^\text{red}_- (\Sigma_g \times S^1, s_k; L(t))$ using $\tau$ and the pairing $(\cdot, \cdot)$ of section 3. Namely, for $\xi, \eta \in HF^\text{red}_-$, we set
\[ (\xi, \eta) = (\tau^{-1}(\xi), \eta) \in L(t) \otimes \mathbb{Z}[H_1(\Sigma_g \times S^1)] L(t) \cong L(t). \]
(Note that $(\cdot, \cdot)$ descends to a pairing on reduced Floer homology with twisted coefficients, by Lemma 3.2. Further, we have that $\Sigma_g \times S^1 \cong -(\Sigma_g \times S^1)$ by conjugation in the $S^1$ factor.) It is straightforward to see that this paring is nondegenerate, in the sense that it provides an identification
\[ HF^\text{red}_- (\Sigma_g \times S^1, s_k; L(t)) \cong \text{Hom}_{L(t)}(HF^\text{red}_- (\Sigma_g \times S^1, s_k; L(t)), L(t)). \]

In particular, for any choice of basis $B_{g,d} = \{ \beta \}$ for the free $L(t)$-module $HF^\text{red}_- (\Sigma_g \times S^1, s_k; L(t)) \cong X(g, d) \otimes L(t)$, we can find a dual basis $B^*_{g,d} = \{ \beta^* \}$ with the property that $(\beta, \gamma^*) = \delta_{\beta \gamma}$ for $\beta, \gamma \in B_{g,d}$.

We have need of some additional algebra to incorporate the action of $H_1(\Sigma_g \times S^1)$ on Floer homology. According to Theorem 6.1, this action is a deformation of the “classical” action (19), which will provide us with sufficient information for our purposes. Let us define
\[ \tilde{X}(g, d) = \bigoplus_{i=0}^{d} \Lambda^i H_1(\Sigma_g) \otimes \mathbb{Z}[U]/U^{d-i+1} \]
as a kind of “dual group” to $X(g, d)$. Then the classical action (19) extends to an action of $\Lambda^* H_1(\Sigma_g)$ on $X(g, d)$ and thence to an action of $\tilde{X}(g, d)$. Furthermore, if we impose a grading on $\tilde{X}(g, d)$ such that $\Lambda^i H_1(\Sigma_g) \otimes U^n$ is homogeneous of degree $i + 2n$ then we see that (19) provides an action of the form
\[ \cdot \cap : \tilde{X}_j(g, d) \otimes X_k(g, d) \longrightarrow X_{k-j}(g, d), \]
where the subscripts indicate the homogeneous parts of degree $j$, $k$ and $k - j$.

In particular, if $\Theta_{g,d}$ denotes a (fixed) generator in lowest degree of $X(g, d)$, then we can define a pairing
\[ \{ \cdot, \cdot \} : \tilde{X}(g, d) \otimes X(g, d) \rightarrow \mathbb{Z} \]
where $\{ x, \xi \}$ is the coefficient of $\Theta_{g,d}$ in $x \cap \xi$. Extending by $L(t)$-linearity, we obtain a pairing
\[ \{ \cdot, \cdot \} : \tilde{X}(g, d) \otimes [X(g, d) \otimes L(t)] \longrightarrow L(t), \]
i.e., an action of $\tilde{X}(g, d)$ on $HF^\text{red}_- (\Sigma_g \times S^1, s_k; L(t))$.

In fact, $\{ \cdot, \cdot \}$ could be defined just as well in terms of the “quantum” action of $\tilde{X}(g, d)$, $x, \xi$. More precisely, the action of $\Lambda(\Sigma_g) = \Lambda^* H_1(\Sigma_g) \otimes \mathbb{Z}[U]$ on $HF^\text{red}_- (\Sigma_g \times S^1, s_k; L(t))$
induces an action of $\tilde{X}(g,d)$ (thought of as a subgroup of $A(\Sigma_g)$), which by Theorem 6.1 is a deformation of the classical action above by lower-degree terms. Since $\Theta_{g,d}$ is in the lowest-degree summand of $HF^{-}_{red}(\Sigma_g \times S^1, \mathfrak{s}_k; L(t))$, we may therefore define $\{x,\xi\}$ to be the coefficient of $\Theta_{g,d}$ in $x,\xi$ (rather than $x \cap \xi$). It is easy to see that $\{\cdot,\cdot\}$ induces an identification

$$\tilde{X}_j(g,d) \cong \text{Hom}(X_{2g-2d-j}(g,d) \otimes L(t), L(t)).$$

Therefore, given a basis $\mathcal{B}_{g,d} = \{\beta\}$ for $X(g,d) \otimes L(t)$ as above, we can find a “quantum dual” basis $\tilde{\mathcal{B}}_{g,d} = \{\tilde{\beta}\}$ for $\tilde{X}(g,d)$ such that

$$\{\tilde{\beta}, \gamma\} = \delta_{\beta\gamma}$$

for $\beta, \gamma \in \mathcal{B}_{g,d}$, which in fact we can take to satisfy

$$\tilde{\beta}.\gamma = \delta_{\beta\gamma} \Theta_{g,d}$$

(thes statements are true only up to units in $L(t)$, unless we assume our basis $\mathcal{B}_{g,d}$ to be induced from a basis of $X(g,d)$ over $\mathbb{Z}$).

The point of all the above is the following:

**Lemma 6.2.** For any basis $\mathcal{B}_{g,d} = \{\beta\}$ for $HF^{-}_{red}(\Sigma_g \times S^1, \mathfrak{s}_k; L(t))$, there exists a quantum dual basis $\tilde{\mathcal{B}}_{g,d} = \{\tilde{\beta}\}$ for $\tilde{X}(g,d)$ with the property that for basis elements $\beta$ and $\gamma$ we have

$$\tilde{\beta}.\gamma = \delta_{\beta\gamma} \Theta_{g,d},$$

where the left-hand side uses the action of $H_1(\Sigma_g)$ (and hence of $\tilde{X}(g,d)$) on Floer homology.

The following was obtained in [2]:

**Theorem 6.3.** Let $\Theta^-$ denote a fixed generator of highest degree in $HF^{-}(S^3)$. Then

$$F^{-}_{\Sigma_g \times D^2, \mathfrak{s}_k}(\Theta^-) = \Xi_k,$$

where for $0 < |k| \leq g - 1$, $\Xi_k$ is a generator (over $L(t)$) in highest degree of $HF^{-}_{red}(\Sigma_g \times S^1, \mathfrak{s}_k; L(t))$ (note that in the topmost degree, this latter module is isomorphic to $L(t)$).

Moreover, for any $\alpha \in A(\Sigma_g)$, we have

$$F^{-}_{\Sigma_g \times D^2, \mathfrak{s}_k}(\Theta^- \otimes \alpha) = \alpha.\Xi_k.$$

Note in particular that $\text{Im}(F^{-}_{\Sigma_g \times D^2})$ is contained in the reduced Floer homology.

**Lemma 6.4.** 1. The lowest-degree generator $\Theta_{g,d}$ for $HF^{-}_{red}(\Sigma_g \times S^1, \mathfrak{s}_k; L(t))$ can be chosen such that

$$(\Theta_{g,d}, \Xi_k) = 1.$$

2. If a homogeneous element $\xi \in HF^{-}_{red}(\Sigma_g \times S^1, \mathfrak{s}_k; L(t))$ does not lie in the lowest nontrivial degree, then

$$(\xi, \Xi_k) = 0.$$

**Proof.** Recall that $HF^\omega(\Sigma_g \times S^1, \mathfrak{s}_k; L(t))$ can be given a relative $\mathbb{Z}$ grading; choose an absolute $\mathbb{Z}$ grading compatible with the sequence relating $HF^{-}$, $HF^\omega$, and $HF^+$. It follows quickly from the definitions that $\langle \cdot, \cdot \rangle$ is nontrivial only on $HF^+_i \otimes HF^{-}_{n-i-2}$ for some fixed integer $n$ depending on the choice of absolute grading (c.f. Proposition 7.11 of [8]), while $\tau$ respects (relative) degree. Hence the second statement follows from the first.
On the other hand, since $(\cdot, \cdot)$ is a duality pairing we know that there exists an $\eta$ such that $(\eta, \Xi_k) \neq 0$. It follows from the remarks above that any element $\xi$ with $\deg(\xi) < \deg(\eta)$ has $(\xi, \cdot) \equiv 0$ for dimensional reasons, violating nondegeneracy of $(\cdot, \cdot)$ unless any such $\eta$ has minimal degree. Again since $(\cdot, \cdot)$ is a duality pairing, we can adjust $\eta$ to ensure that $(\eta, \Xi_k) = 1$, which proves the first claim.

\[ \square \]

### 6.2. Calculating the Relative Invariant.

Suppose now that $M$ is a closed 4-manifold $(b^+(M) \geq 2)$ containing $\Sigma_g$, embedded with trivial normal bundle, and $\mathfrak{s}$ is a spin$^c$ structure on $M$ with $\langle c_1(\mathfrak{s}), [\Sigma_g] \rangle = 2k$, $0 < |k| \leq g - 1$. Then the results of section 5 apply to show that we can use the splitting $M = Z \cup_{\Sigma_g \times S^1} \Sigma_g \times D^2$ to calculate $\Phi_{M, \mathfrak{s}}$ using a pairing on the Floer homology of $\Sigma_g \times S^1$:

\[ \Phi_{M, \mathfrak{s}-t} \langle \alpha \rangle = \langle \tau^{-1} \Psi_{M, \mathfrak{s}}(\alpha_1), \Psi_{\Sigma_g \times D^2, \mathfrak{s}_k}(\alpha_2) \rangle. \]

Indeed, we must have $b^+(Z) \geq 1$, while we saw previously that $\Psi_{\Sigma_g \times D^2}$ takes values in the reduced Floer homology, so that Theorem 5.7 applies. Our goal here is in a sense to “invert” the above equation, i.e., understand the relative invariant $\Psi_{M, \mathfrak{s}}$ in terms of the invariants $\Phi_{M, \mathfrak{s}}$ of the closed 4-manifold.

Now, the pairing on the right hand side of the previous expression takes values in

\[ \mathbb{Z}[K(Z)] \otimes_{[H^1(\Sigma_g \times S^1)]} \mathbb{Z}[K(\Sigma_g \times D^2)] \cong \mathbb{Z}[K(M, \Sigma_g \times S^1)] \]

The isomorphism $f$ indicated above (c.f. Lemma 2.7) factors through the map

\[ \rho \otimes \sigma : \mathbb{Z}[K(Z)] \otimes [K(\Sigma_g \times D^2)] \longrightarrow \mathbb{Z}[K(M, \Sigma_g \times S^1)] \otimes \mathbb{Z}[K(M, \Sigma_g \times S^1)] \cong \mathbb{Z}[K(M, \Sigma_g \times S^1)] \]

where $\rho$ is induced by the natural projection

\[ \rho : K(Z) \cong \frac{H^1(\Sigma_g \times S^1)}{H^1(Z)} \longrightarrow \frac{H^1(\Sigma_g \times S^1)}{H^1(Z) + H^1(\Sigma_g \times D^2)} \cong K(M, \Sigma_g \times S^1) \]

and $\sigma$ is induced by

\[ \sigma : K(\Sigma_g \times D^2) \cong \frac{H^1(\Sigma_g \times S^1)}{H^1(\Sigma_g \times D^2)} \longrightarrow \frac{H^1(\Sigma_g \times S^1)}{H^1(Z) + H^1(\Sigma_g \times D^2)} \cong K(M, \Sigma_g \times S^1). \]

In general the latter is a surjection of $K(\Sigma_g \times D^2) \cong <PD[\Sigma_g]> \cong \mathbb{Z}$ onto $K(M, \Sigma_g \times S^1) \cong <PD[\Sigma_g]>$. If $\Sigma_g$ represents a non-torsion class in $H_2(M; \mathbb{Z})$, then $\sigma$ is an isomorphism.

Now, the kernel of the map $K(Z) \longrightarrow K(M, \Sigma_g \times S^1)$ above corresponds to the image of $H^1(\Sigma_g \times D^2)$ in $H^1(\Sigma_g \times S^1)$. Under Poincaré duality, this subspace is spanned by “rim tori,” namely those tori of the form $\gamma \times S^1 \subset \Sigma_g \times S^1$ where $\gamma$ is an essential circle on $\Sigma_g$. Therefore, on the level of group rings the homomorphism $\rho_\star : \mathbb{Z}[K(Z)] \rightarrow \mathbb{Z}[K(M, \Sigma_g \times S^1)]$ can be understood as substituting 1 for all variables corresponding to rim tori. More explicitly, we have

\[ \rho_\star \left( \sum_{\epsilon_i \in K(Z)} m_i \epsilon_i \right) = \sum_{\epsilon_j \in K(M, \Sigma_g \times S^1)} n_j \epsilon_j, \quad \text{where} \quad n_j = \sum_{\epsilon_i \in \rho^{-1}(\epsilon_j)} m_i. \]

We now express the relative invariant $\rho_\star(\Psi_{Z, \mathfrak{s}}(\alpha_1))$ in terms of a basis of the Floer homology of $\Sigma_g \times S^1$. From now on we assume...
Under these hypotheses, we have that $g_L$ above), where $\Sigma$ according to Theorem 6.1.

Of this module can be expressed in terms of this basis as a combination $\xi$ using the quantum dual basis $\beta$ according to Lemma 6.2 and Lemma 6.4, the coefficients $n_{\beta}$ are given by $n_{\beta} = (\tilde{\beta}, \xi, \Xi_k)$, using the quantum dual basis $\{\tilde{\beta}\}$. Applying this idea in the case that $\xi = \rho_s(\Psi_{Z,s}(\alpha_1))$, we obtain the following:

**Proposition 6.5.** For $M$ a closed 4-manifold containing an embedded surface $\Sigma_g$ with trivial normal bundle, let $Z = M \setminus (\Sigma_g \times D^2)$. Let $s$ be a spin$^c$ structure on $M$, and assume that hypotheses (1), (2), and (3) above are satisfied. Then for any $\alpha_1 \in \mathbb{A}(Z)$, the relative invariant $\rho_s(\Psi_{Z,s}(\alpha_1))$ (summed over rim tori, and using the same symbol for $s$ and its restriction to $Z$) is expressed in terms of a basis for Floer homology by a linear combination of basis elements whose coefficients are absolute invariants of the manifold $M$. Explicitly, we have for a basis $B_{g,d} = \{\beta\}$ as above with quantum dual basis $\{\tilde{\beta}\}$,

$$\rho_s(\Psi_{Z,s}(\alpha_1)) = \sum_{\beta \in B_{g,d}} \Phi_{M,s-\ell}(\alpha_1 \otimes \tilde{\beta}) \cdot \beta \otimes e^t.$$  

(23)

Here $\alpha_1 \otimes \tilde{\beta}$ denotes the image of that element under the inclusion-induced map $\mathbb{A}(Z) \otimes X(g, d) \to \mathbb{A}(M)$.

When $b^+(M) = 1$, the invariant $\Phi_{M,s}$ is calculated with respect to $v = [\Sigma_g]$.

**Proof.** As observed at the beginning of this subsection, we have that for a spin$^c$ structure $s$ on $M$ and for $\alpha \in \mathbb{A}(M)$,

$$\sum_{t \in K(M, \Sigma_g \times S^1)} \Phi_{M, s-t}(\alpha) \cdot e^t = (\Psi_{Z,s}(\alpha_1), \Psi_{\Sigma_g \times D^2, s}(\alpha_2))$$  

(24)

where $k$ is determined by condition (2) above. Here $\alpha_1 \in \mathbb{A}(Z)$ and $\alpha_2 \in \mathbb{A}(\Sigma_g \times D^2)$ satisfy $\alpha_1 \otimes \alpha_2 = \alpha$.

Expressing the relative invariant $\rho_s(\Psi_{Z,s}(\alpha_1))$ in terms of the basis $\beta$ as indicated previously gives

$$\rho_s(\Psi_{Z,s}(\alpha_1)) = \sum_{\beta \in B_{g,d}} (\tilde{\beta}, \rho_s(\Psi_{Z,s}(\alpha_1)), \Xi_k) \cdot \beta$$

$$= \pm \sum_{\beta \in B_{g,d}} (\rho_s(\Psi_{Z,s}(\alpha_1)), \tilde{\beta}, \Xi_k) \cdot \beta.$$
where the sign depends on the degree of $\tilde{\beta}$. When pairing with an element of $HF^{-}_{\text{red}}(\Sigma_{g} \times S^{1}, s_{k}; L(t))$, the homomorphism $\rho_{*}$ has no effect (it merely sums over rim tori, which are trivial in $M$). Thus we can write the last line above as

$$\sum_{\beta \in B_{g,d}} (\Psi_{Z_{d}}(\alpha_{1}), \Psi_{\Sigma_{g} \times D^{2}, s_{k}}(\tilde{\beta})) \cdot \beta,$$

using Theorem 6.3. The expression above is (up to sign and multiplication by a unit in $L(t)$) just the right hand side of (23), according (24).

### 6.3. Proof of Product Formula: $g \geq 2$. With Proposition 6.5 in hand the product formula for a general fiber sum follows quickly. Recall that for $i = 1, 2$ we are given closed 4-manifolds $M_{i}$ with embeddings $f_{i} : \Sigma \to M_{i}$ of a genus $g$ surface into $M_{i}$ with trivial normal bundle, and spin$^{c}$ structures $s_{i}$ on $M_{i}$ satisfying $\langle c_{1}(s_{i}), [\Sigma] \rangle = 2k$ for some $k$ with $0 < |k| \leq g - 1$ (independent of $i$). Here we assume $g \geq 2$. Let $Z_{i} = M_{i} \setminus \Sigma \times D^{2}$ (omitting the embeddings $f_{i}$ from the notation). We write $X = M_{1} #_{\Sigma} M_{2}$ for the fiber sum along $\Sigma$, and $s$ for some fixed spin$^{c}$ structure on $X$ satisfying $s|_{Z_{i}} = s_{i}|_{Z_{i}}$. Note that $X$ contains a natural copy of $\Sigma \times S^{1}$, under the identification $X = Z_{1} \cup_{\Sigma \times S^{1}} Z_{2}$ of smooth manifolds.

Let $R$ denote the subspace of $H^{2}(X; \mathbb{Z})$ spanned by classes Poincaré dual to rim tori, and similarly write $R_{i}, i = 1, 2$ for the subspace of $H^{2}(Z_{i}, \partial Z_{i}; \mathbb{Z})$ spanned by the Poincaré duals of rim tori in $Z_{i}$. Then we have a commutative diagram

$$
\begin{array}{cccc}
K(Z_{1}) \oplus K(Z_{2}) & \xrightarrow{\rho_{1} \oplus \rho_{2}} & K(M_{1}, \Sigma \times S^{1}) \oplus K(M_{2}, \Sigma \times S^{1}) \\
\downarrow & & \downarrow \\
K(X, \Sigma \times S^{1}) & \xrightarrow{\rho_{X}} & K(X, \Sigma \times S^{1}) / R
\end{array}
$$

(25)

where $\rho_{i}$ are the projections defined in (22) and $\rho_{X}$ is the obvious quotient.

**Theorem 6.6** (Fiber sums along surfaces of genus $g \geq 2$). Assume that $b^{+}(M_{i}) \geq 1$ for $i = 1, 2$. For $X = M_{1} #_{\Sigma} M_{2}$ as above, we have for any $\alpha \in A(X)$,

$$\sum_{t \in K(X, \Sigma \times S^{1})} \Phi_{X,s}^{-1}(\alpha) \cdot e[t] = \sum_{\beta \in B_{g,d}} \Phi_{M_{1},s_{1}-t_{1}}(\alpha_{1} \otimes \tilde{\beta}) \Phi_{M_{2},s_{2}-t_{2}}(\alpha_{2} \otimes \tilde{\beta}^{*}) \cdot e[t_{1}-t_{2}]$$

up to signs and multiplication by an element of $K(X, \Sigma \times S^{1}) / R$.

In the expression above, $[t]$ denotes the equivalence class of $t \in K(X, \Sigma \times S^{1})$ modulo $R$, while $[t_{1}-t_{2}]$ denotes the image of $(t_{1}, -t_{2})$ under the map

$$K(M_{1}, \Sigma \times S^{1}) \oplus K(M_{2}, \Sigma \times S^{1}) \to K(X, \Sigma \times S^{1}) / R.$$

Further, $\alpha_{1} \in A(Z_{1})$ and $\alpha_{2} \in A(Z_{2})$ are any elements satisfying $\alpha_{1} \otimes \alpha_{2} \leftrightarrow \alpha$ under $A(Z_{1}) \otimes A(Z_{2}) \to A(X)$, and $B_{g,d}$ is any basis for $HF^{-}_{\text{red}}(\Sigma \times S^{1}, s_{k} ; L(t))$ as an $L(t)$-module. The set $\{\tilde{\beta}\}$ indicates the “quantum dual” basis to $B_{g,d}$ of the group $\tilde{X}(g, d) \cong H_{*}(\text{Sym}^{d}(\Sigma))$. 


Likewise, \( \{ \beta^* \} \) is the basis dual to \( \{ \beta \} \) with respect to the pairing \((\cdot, \cdot)\) on Floer homology described in section 6.1, equation (20), and \( \{ \beta^* \} \) is the corresponding basis of \( X(g,d) \).

As usual, the invariants for manifolds with \( b^+ = 1 \) are calculated with respect to the summing surface.

**Proof.** We simply apply the product formula, Theorem 5.7, to the splitting \( X = Z_1 \cup_{\Sigma \times S^1} Z_2 \) and apply Proposition 6.5 to express the relative invariants of \( Z_1 \) and \( Z_2 \) in terms of absolute invariants of \( M_1 \) and \( M_2 \). The only slight annoyance is the necessity of summing over rim tori.

Applying \( \rho_X \) to the left hand side of (15) and \( \rho_{1*} \otimes \rho_{2*} \) to the right, we obtain

\[
\sum_{t \in K(X,\Sigma \times S^1)} \Phi_{X,\sigma-t}(\alpha) \cdot e[\ell] = (\rho_{1*} \Psi_{Z_1,\sigma_1}(\alpha_1), \rho_{2*} \Psi_{Z_2,\sigma_2}(\alpha_2)) = \sum_{\beta \in B_{g,d}} \Phi_{M_1,\sigma_1-t_1}(\alpha_1 \otimes \tilde{\beta}) \Phi_{M_2,\sigma_2-t_2}(\alpha_2 \otimes \tilde{\gamma})(\beta, \gamma) \cdot e[t_1-t_2].
\]

Here we make use of two bases \( B_{g,d} = \{ \beta \} \) and \( B'_{g,d} = \{ \gamma \} \) for \( HF_{\text{red}}^-(\Sigma \times S^1, s_k; L(t)) \), and use the identification of coefficient modules supplied by Lemma 2.7. We may assume that \( B_{g,d} \) and \( B'_{g,d} \) are dual bases with respect to \((\cdot, \cdot)\), which immediately gives (26).

**Proof of Theorem 1.2.** As remarked previously, \( K(M_i, \Sigma \times S^1) \) is infinite cyclic, generated by the Poincaré dual of \( \Sigma \) in \( M_i \), for each of \( i = 1, 2 \). Hence Theorem 6.6 implies that for a spin\(^c\) structure \( s \in Spin^c(X) \) with \( \langle c_1(s), [\Sigma] \rangle = 2k \) we have

\[
\Phi_{X,s}^{\text{rim}}(\alpha) = \sum_{\beta \in B_{g,d}} \Phi_{M_1,\sigma_1+n_1s}(\alpha_1 \otimes \tilde{\beta}) \Phi_{M_2,\sigma_2+n_2s}(\alpha_2 \otimes \tilde{\beta}^*)
\]

for some integer \( n_0 \), where \( s_1 \) and \( s_2 \) are fixed spin\(^c\) structures whose restrictions to the complement of \( \Sigma \times D^2 \) agree with those of \( s \), and that have \( \langle c_1(s_i), [\Sigma] \rangle = 2k \). Here \( S \) denotes the Poincaré dual of \([\Sigma]\) in the appropriate 4-manifold, while \( \Phi_{X,s}^{\text{rim}} \) indicates the sum of the invariants over all spin\(^c\) structures of the form \( s + h \) for \( h \in \mathcal{R} \). The above is equivalent to the formula of Theorem 1.2.

**Proof of Theorem 1.1.** The issue we must address is the correspondence of spin\(^c\) structures on the summands \( M_1, M_2 \) with those on \( X \). In effect, this is equivalent to the determination of the integer \( n_0 \) in Theorem 1.2, which may depend \textit{a priori} on \( s \) and \( \alpha \).

Recall that the product formula (3) of Theorem 1.1 is defined using a multiplication between elements of group rings based on a “patching” construction. We find it easiest to describe this construction in homology rather than cohomology; the cohomological version is obtained by Poincaré duality. Suppose, then, that \( x_1 \in H_2(M_i) \) and \( x_2 \in H_2(M_2) \) are integral homology classes, represented by embedded surfaces also denoted \( x_1, x_2 \), and assume that \( x_i, \Sigma_i = m \) for \( i = 1, 2 \). Let \( \rho : H_2(M_i) \to H_2(Z_i, \partial Z_i) \) denote the composition of the natural map \( H_2(M_i) \to H_2(M_i, \Sigma_i \times D^2) \) followed by the excision isomorphism of the latter.
group with $H_2(Z_i, \partial Z_i)$ where $Z_i = M_i \setminus \text{int}(\Sigma_i \times D^2)$. Consider the long exact sequence for $X = Z_1 \cup_{\partial} Z_2$:

$$\cdots \to H_2(\Sigma \times S^1) \to H_2(X) \to H_2(Z_1, \partial Z_1) \oplus H_2(Z_2, \partial Z_2) \to H_1(\Sigma \times S^1) \to \cdots$$

The condition on $x_i, \Sigma_i$ and the fact that the $\rho(x_i)$ are restrictions of classes on the closed manifolds $M_i$ imply that there exists a lift $x \in H_2(X)$ of $(\rho(x_1), -\rho(x_2))$, uniquely determined up to the image of $H_2(\Sigma \times S^1)$.

Choose the surfaces $x_i$ to intersect $\Sigma_i \times D^2$ in a collection of normal disks; at the expense of increasing the genus of the $x_i$ we may assume that there are exactly $|m|$ such disks. Then removing $\Sigma_i \times D^2$ from each of $M_1, M_2$ and gluing we can obtain a smooth surface representing the lifted class $x$. It is clear that $x$ has $x, \Sigma = m$, and furthermore by using pushoffs of the $x_i$ that are disjoint from the normal disks in $\Sigma_i \times D^2$ we see that the self-intersection of $x$ satisfies $x^2 = x_1^2 + x_2^2$.

Now let $x_1 \ast x_2 = x + 2\varepsilon \Sigma$, where $\varepsilon$ is the sign of $m$. Then the self-intersection of $x_1 \ast x_2$ is

$$(x_1 \ast x_2)^2 = x_1^2 + x_2^2 + 4|m|,$$

and moreover the class $x_1 \ast x_2$ is determined by this condition up to addition of elements of $H_2(\Sigma \times S^1)/[\Sigma]$, in other words, up to rim tori. The (Poincaré dual of the) assignment

$$(x_1, x_2) \mapsto x_1 \ast x_2 \in H_2(X; \mathbb{Z})/\mathcal{R}$$

gives rise to the multiplication used on the right hand side of (3).

We now turn to the deduction of Theorem 1.1 from Theorem 1.2.

We may assume that the basis $\overline{\beta}$ for $\tilde{X}(g, d)$ consists of homogenous elements of the form $\omega \wedge U^i$, where $\omega \in \Lambda^3 H_1(\Sigma)$. In [10], Ozsváth and Szabó showed that when $|k| > \frac{1}{3}(g - 1)$ (equivalently, when $3d < 2g - 1$) the action of $H_1(\Sigma \times S^1)$ on $HF^+ = HF_{red}$ is given by the cap product (19) exactly, not just to leading order. In this case it is easy to check that $\overline{\beta}^r$ is also homogeneous, and if $\beta$ has degree $n$ in $\tilde{X}(g, d)$ then $\overline{\beta}^r$ has degree $2d - n$. Hence in (27) we have $\deg(\overline{\beta}) + \deg(\overline{\beta}^r) = 2d = 2g - 2 - 2|k|$, yielding

$$\deg(\alpha_1 \otimes \overline{\beta}) + \deg(\alpha_2 \otimes \overline{\beta}^r) = \deg(\alpha) + 2g - 2 - 2|k|.$$ 

On the other hand, if $\Phi_{X, r}(\xi) \neq 0$ we must have $d(\tau) = \deg(\xi)$ (c.f. (1)). Making this substitution in the above equation gives

$$c_1^2(s) = c_1^2(s_1 + n_1 S) + c_1^2(s_2 + n_2 S) + 8|k|.$$ 

Therefore (27) can be rephrased as follows: fix spin$^c$ structures $\tau_i$ on $M_i$ and $s$ on $X$, restricting compatibly to $Z_i$ and satisfying $c_1^2(s) = c_1^2(\tau_1) + c_1^2(\tau_2) + 8|k|$; then

$$\Phi_{X, s}^{Rim}(\alpha) = \sum_{\begin{array}{c}
\beta \in B_{g, d} \\
n_1, n_2 \in \mathbb{Z}
\end{array}} \Phi_{M_1, r_1 + n_1 S}(\alpha_1 \otimes \overline{\beta}) \Phi_{M_2, r_2 + n_2 S}(\alpha_2 \otimes \overline{\beta}^r)$$

where the sum is over integers $n_1, n_2$ such that

$$c_1^2(s) = c_1^2(\tau_1 + n_1 S) + c_1^2(\tau_2 + n_2 S) + 8|k|.$$ 

This translates quickly to the condition that $n_1 + n_2 = 0$, which justifies the product formula in Theorem 1.1. 

$\square$
6.4. Proof of Product Formula: $g = 1$. For a closed 4-manifold $M$ containing a square-0 torus $\Sigma$, we write $M = Z \cup_{\Sigma \times S^1} (\Sigma \times D^2)$ and as before are interested in expressing the relative invariant $\Psi_{Z,s}$ in terms of absolute invariants of $M$. We assume from now on that the image of $H^1(Z;\mathbb{Z})$ in $H^1(\Sigma \times S^1;\mathbb{Z})$ is trivial: this is the same as the requirement that $K(Z) = H^1(\Sigma \times S^1;\mathbb{Z})$, i.e., that the relative invariant of $Z$ takes values in the fully-twisted Floer homology of $\Sigma \times S^1$.

Note that according to the adjunction inequality for Ozsváth-Szabó invariants [8], we have $\Phi_{M,s} \equiv 0$ unless $\langle c_1(s), \Sigma \rangle = 0$. Thus in the following we assume that this pairing vanishes.

It was shown in [9] that the Floer homology of $\Sigma \times S^1$ in the torsion spin$^c$ structure and with fully-twisted coefficients is given by:

$$HF^+_k(\Sigma \times S^1; s_0; R_{\Sigma \times S^1}) = \begin{cases} 
0 & k \equiv 3/2 \mod \mathbb{Z}, k \geq 3/2 \\
\mathbb{Z} & k \equiv 1/2 \mod \mathbb{Z}, k \geq 1/2 \\
\ker \epsilon & k = -1/2 
\end{cases}$$

Here $\epsilon : R_{\Sigma \times S^1} = \mathbb{Z}[H^1(\Sigma \times S^1)] \to \mathbb{Z}$ is the “augmentation” that maps every element of $H^1(\Sigma \times S^1)$ to 1 and $\ker \epsilon$ is the “augmentation ideal.” In particular, $HF^+_r(\Sigma \times S^1; s_0, R_{\Sigma \times S^1}) = \ker \epsilon$, supported in degree $-1/2$.

Choosing a basis $\{r, s, t\}$ for $H^1(\Sigma \times S^1; \mathbb{Z})$ provides an identification of $R_{\Sigma \times S^1}$ with the Laurent polynomial ring $L(r,s,t)$ in three variables, and in this notation $\ker \epsilon$ is generated by the elements $r-1$, $s-1$ and $t-1$. Here as before, we take $t$ to be the Poincaré dual of the torus $\Sigma \times \{pt\}$, and $r$ and $s$ to be dual to rim tori.

The following was obtained in [2].

**Proposition 6.7.** The homomorphism

$$F^+_{\Sigma \times D^2} : HF^+(\Sigma \times S^1; s_0; R) \to HF^+(S^3; L(t)) = HF^+(S^3; \mathbb{Z}) \otimes L(t)$$

induced by the cobordism $(\Sigma \times D^2) \setminus B^4$ factors through $HF^+_r$. Furthermore, identifying $HF^+_r(\Sigma \times S^1; s_0; R) \cong \ker \epsilon \subset R$, the homomorphism $F^+_{\Sigma \times D^2}$ has image in the lowest non-trivial degree and is given by

$$a(r, s, t) \mapsto \frac{a(1,1,t)}{t-1} \in HF^+_0(S^3;L(t)) \cong L(t).$$

As in the higher-genus situation, it will be convenient to project from the fully-twisted Floer homology of $\Sigma \times S^1$ to that with coefficients in $L(t)$. As before, we denote by $\rho : R_{\Sigma \times S^1} \to L(t)$ the homomorphism $a(r, s, t) \mapsto a(1,1,t)$ induced by the projection of $H^1(\Sigma \times S^1; \mathbb{Z})$ onto the dual of $[\Sigma]$, and write $\rho_*$ for the induced map in Floer homology. We recall the structure of the Floer homology of $\Sigma \times S^1$ with $L(t)$ coefficients.

**Proposition 6.8** ([2]). The Heegaard Floer homology $HF^+(\Sigma \times S^1; s_0; L(t))$ of the 3-torus $\Sigma \times S^1$ in the torsion spin$^c$ structure, with coefficients in the group ring of the subgroup of $H^1(\Sigma \times S^1; \mathbb{Z})$ spanned by the Poincaré dual of $\Sigma$, is given by

$$HF^+_k(\Sigma \times S^1; s_0; L(t)) = \begin{cases} 
\mathbb{Z}^2 & k \equiv 1/2 \mod \mathbb{Z}, k \geq 1/2 \\
L(t) \oplus \mathbb{Z}^2 & k = -1/2 \\
0 & \text{otherwise} 
\end{cases}$$

In particular, $HF^+_r(\Sigma \times S^1; s_0; L(t))$ is isomorphic to $L(t)$, supported in degree $-1/2$. 


While it is not important for our purposes, the $L(t)$-module structure on $HF_+^k(\Sigma \times S^1, s_0; L(t))$ is not trivial for $k > 0$: see [2] for details.

**Lemma 6.9.** The map induced by $\rho$ in reduced Floer homology can be identified with the homomorphism

$$
\rho_* : HF^+_\text{red}(\Sigma \times S^1, s_0; R_{\Sigma \times S^1}) \rightarrow HF^+_\text{red}(\Sigma \times S^1, s_0; L(t))
$$

\[ \ker \epsilon \rightarrow L(t) \]

(28)

\[ a(r, s, t) \leftrightarrow a(1, 1, t). \]

Proof. As an abstract $R_{\Sigma \times S^1}$-module, $\ker \epsilon \otimes L(t)$ is isomorphic to $HF^+_{-1/2}(\Sigma \times S^1, s_0; L(t)) = \mathbb{Z} \oplus \mathbb{Z} \oplus L(t)$. However, we claim that the induced homomorphism $\rho_*$ on $HF^+_\text{red}$ factors through the natural inclusion $\ker \epsilon \hookrightarrow R_{\Sigma \times S^1}$, followed by the map $R_{\Sigma \times S^1} \rightarrow L(t)$ indicated in (28) (indeed, this is used implicitly in the expression (28), where an element of $\ker \epsilon$ is written as a 3-variable Laurent polynomial).

To see the claim, we examine the surgery sequence connecting $M(0, 0, \infty)$, $\Sigma \times S^1 = M(0, 0, 0)$, and $M(0, 0, 1)$. Here we use the notation of [9], where $M(a, b, c)$ denotes the 3-manifold obtained by performing surgery on the Borromean rings with coefficients $a$, $b$, $c$.

Writing $R$ for $R_{\Sigma \times S^1}$, we have the diagram

$$
\xymatrix{
HF^+_{-1/2}(M(0, 0, 0), R) & HF^+_{-1}(M(0, 0, 1), R) & HF^+_{-1}(M(0, 0, \infty), R) \\
\rho_* & \rho_* & \rho_*
}
$$

It was shown in [9] that the top row of this diagram can be identified with

$$
0 \rightarrow \ker \epsilon \rightarrow L(t) \oplus R \rightarrow L(t) \rightarrow 0,
$$

inducing an inclusion of $\ker \epsilon$ into $R$. The bottom row of the diagram appears as

(29)

$$
\mathbb{Z}^2 \oplus L(t) \rightarrow L(t) \oplus L(t) \rightarrow L(t) \rightarrow 0.
$$

The reduced Floer homology $HF^+_{\text{red}}(M(0, 0, 0), L(t))$ is supported in degree $-1/2$ and is identified with the copy of $L(t)$ in the first term above. Furthermore, it is not hard to see that the kernel of the first map in (29) is the $\mathbb{Z}^2$ summand. The lemma follows.

The analogue of Proposition 6.5 in the genus 1 case is:

**Proposition 6.10.** Suppose $M$ is a closed 4-manifold with $b^+(M) \geq 2$ containing an embedded torus $\Sigma$ with trivial normal bundle representing a non-torsion element of $H_2(M; \mathbb{Z})$ and let $Z = M \setminus (\Sigma \times D^2)$. Let $s$ be a spin$^c$ structure on $M$ satisfying $\langle c_1(s), \Sigma \rangle = 0$. Suppose further that the restriction $H^1(Z; \mathbb{Z}) \rightarrow H^1(\Sigma \times S^1; \mathbb{Z})$ is trivial. Then for any $\alpha \in \mathbb{A}(Z)$, the relative invariant $\rho_*(\Psi_{Z,s}(\alpha)) \in HF^+_{\text{red}}(\Sigma \times S^1, s_0; L(t))$ is related to the absolute invariants of $M$ by the formula

$$
\frac{1}{t - 1} \rho_*(\Psi_{Z,s}(\alpha)) = \sum_{n \in \mathbb{Z}} \Phi_{M,s+nPD[\Sigma]}(\alpha) \cdot t^n
$$
under the identification \( HF^{-}_\text{red}(\Sigma \times S^1, s_0; L(t)) \cong L(t) \), and the inclusion-induced map \( \Lambda(Z) \to \Lambda(M) \).

**Proof.** Since the map induced by \( \Sigma \times D^2 \) in \( HF^+ \) factors through reduced homology, we have

\[
\sum_{n \in \mathbb{Z}} \Phi_{M, s}(\alpha) \cdot t^n = \langle F^+_{\Sigma \times D^2} \tau^{-1} F^-_{Z, s}(\alpha \otimes \Theta^-), \Theta^- \rangle = F^+_{\Sigma \times D^2} \tau^{-1} F^-_{Z, s}(\alpha \otimes \Theta^-),
\]

where on the right we are identifying an element of \( HF^+_0(S^3; L(t)) \) with a Laurent polynomial. Combining Proposition 6.7 with Lemma 6.9 gives the result, recalling that \( \Psi_{Z, s}(\alpha) \) is simply \( F^-_{Z, s}(\alpha \otimes \Theta^-) \).

The product formula follows easily now.

**Theorem 6.11** (Fiber sums along tori). Let \( X = M_1 \#_\Sigma M_2 \) be the result of the fiber sum of two 4-manifolds \( M_1, M_2 \) along embeddings \( f_i : \Sigma \to M_i, i = 1, 2 \) of tori, satisfying the assumptions of Proposition 6.10, and such that the induced classes \([\Sigma_1] \) and \([\Sigma_2] \) agree. Let \( s_i \) be spin\(^c\) structures on \( M_i \) with \( \langle c_1(s_i), [\Sigma] \rangle = 0 \) for \( i = 1, 2 \), and suppose \( s \) is a spin\(^c\) structure on \( X \) restricting to \( s_i|_{Z_i} \) on each \( Z_i \). Then for any \( \alpha \in \Lambda(X) \),

\[
\sum_{t \in K(X, \Sigma \times S^1)} \Phi_{X, s-t}(\alpha) \cdot e^t = (T - 1)(T^{-1} - 1) \sum_{t_1 \in K(M_1, \Sigma \times S^1)} \Phi_{M_1, s_1-t_1}(\alpha_1) \Phi_{M_2, s_2-t_2}(\alpha_2) \cdot e^{t_1-t_2}
\]

up to signs and multiplication by an element of \( K(X, \Sigma \times S^1)/\mathcal{R} \).

Here \( T \in \mathbb{Z}[K(X, \Sigma \times S^1)] \) denotes the element corresponding to the Poincaré dual of the torus \( \Sigma_i \) induced from \( M_i \), while the remaining notation is as in Theorem 6.6.

**Proof.** We apply the product formula (15) to \( X = Z_1 \cup Z_2 \):

\[
\sum_{t \in K(X, \Sigma \times S^1)} \Phi_{X, s-t}(\alpha) \cdot e^t = (\Psi_{Z_1, s_1}(\alpha_1), \Psi_{Z_2, s_2}(\alpha_2)).
\]

Applying \( \rho_1 \times \rho_2 \) to both sides yields

\[
\sum_{t \in K(X, \Sigma \times S^1)} \Phi_{X, s-t}(\alpha) \cdot e^t = (\rho_1(s_1(\alpha_1)), \rho_2(s_2(\alpha_2)))
\]

\[
= \left( (T - 1) \sum_{t_1 \in K(M_1, \Sigma \times S^1)} \Phi_{M_1, s_1-t_1}(\alpha_1) \cdot e^{t_1},
\right)
\]

\[
\left( (T - 1) \sum_{t_2 \in K(M_2, \Sigma \times S^1)} \Phi_{M_2, s_2-t_2}(\alpha_2) \cdot e^{t_2} \right),
\]

according to Proposition 6.10, since the variable \( t \) in that proposition corresponds here to the generator of \( K(M_i, \Sigma \times S^1) \cong \mathbb{Z} \), which we denote by \( T \).

The factors of \( (T - 1) \) pull out of the pairing to give \( (T - 1)(T^{-1} - 1) \) since \( \langle \cdot, \cdot \rangle \) is antilinear in the second position. The pairing is taking place in \( HF^{-}_\text{red}(\Sigma \times S^1, s_0; L(t)) \), which is a cyclic \( R_{\Sigma \times S^1} \)-module, and hence gives rise to the product of polynomials as indicated in the statement of the theorem.
Proof of Theorem 1.5. This is now a straightforward calculation along the lines of the proof of Theorem 1.1, using a comparison of the indices $d(s)$ as before.
References


